SDE | 2010

XXVII Seminar in Differential Equations
Bořetice, May 31 - June 4, 2010

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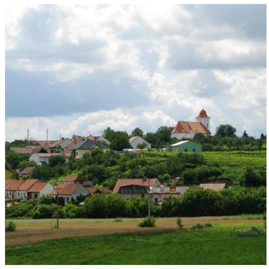

Optimality of Function Spaces in Sobolev Embeddings

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Bořetice, Czech Republic
48º55´9.955"N, 16º50´46.676"E

FACULTY
OF APPLIED SCIENCES
UNIVERSITY
OF WEST BOHEMIA

sde.kma.zcu.cz/2010

Proceedings of Seminar in Differential Equations[1]

Volume II

Optimality of function spaces in Sobolev embeddings[2]

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Abstract: We study optimality of various types of function spaces that appear in all kinds of Sobolev embeddings. We focus in particular on rearrangement-invariant Banach function spaces. We construct optimal target spaces, optimal domain spaces, and optimal pairs of function spaces in Sobolev embeddings. The methods and results apply also to various related topics such as boundary traces problem, logarithmic Sobolev inequalities or compactness of Sobolev embeddings.

2000 Mathematics Subject Classification: 46E35

Keywords: Sobolev spaces, rearrangement-invariant spaces, optimal range, optimal domain, reduction theorems, interpolation, duality, compact embeddings, fundamental functions, boundary traces, Gaussian embeddings.

Bořetice
May 31 – June 4, 2010

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[1]Seminar was supported by MŠMT ČR – ME09109.

[2]This research was supported by the grant 201/08/0383 of the Grant Agency of the Czech Republic.

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ISBN 978-80-261-0186-4

1. Prologue

In their initial definition, Sobolev spaces are certain specific Banach spaces containing weakly differentiable functions of several variables that arise in connection with problems in mathematical physics and PDEs. Their key importance for solving such problems has been known since about late 1930's when the first contribution of S.L. Sobolev to their study appeared ([106]). Sobolev spaces constitute an indispensable tool in applications, but they are also quite interesting on their own, as a very particular mathematical structure with unique properties. Since their first appearance, important monographs dedicated to them appeared (see e.g. [1, 4, 83, 84, 73]) and a lot of information about them became known. However, there is still a lot to do in their investigation, and it seems that recently the interest in Sobolev spaces is growing.

The most important feature of Sobolev spaces is how they embed (or how they compactly embed) into other function spaces, primarily into Lebesgue spaces and their generalizations. Lebesgue spaces of course play a most decisive role in analysis but there are other function spaces which are also of interest. In certain situations the class of Lebesgue spaces is not rich enough to enable one to describe all the important characteristics precisely enough, and in these situations other function spaces are called into the play. Depending on the nature of the problem, the next call usually goes for either Orlicz spaces or for Lorentz spaces. Orlicz spaces constitute a very convenient replacement for Lebesgue spaces especially in cases when certain limiting growth is needed, either more rapid or more slow than the power functions can offer. Lorentz spaces and their likes, on the other hand, turn out to be a very precise tool for fine-tuning and tightening of the results.

Our main aim in this text is to carry out a thorough study of sharpness, or optimality, of function spaces involved in Sobolev embeddings. This sharpness, of course, depends on the context within which it is considered. What is sharp in the class of Lebesgue spaces need not be sharp any more in the environment of Orlicz or Lorentz spaces. However it can be demonstrated with examples that each of these classes of function spaces also has its own specific limitations. Therefore, we shall study the optimality of function spaces within a quite broad and fairly general category of the so-called rearrangement-invariant spaces, which provides a common roof for Lebesgue, Orlicz and Lorentz spaces, and also for many more, such as Zygmund classes, Lorentz–Zygmund spaces, classical Lorentz spaces, endpoint Lorentz and Marcinkiewicz spaces, the spaces of Maz'ya–Hansson–Brézis–Wainger type, and more.

2. Introduction

The principal question that arises in connection with the Sobolev spaces is the following one: given an information about the gradient of a scalar function of several real variables, what can we say about the function itself? For example, if the gradient belongs to a certain Lebesgue space, will the function itself belong to the same space? Or to a different Lebesgue space? And, if there are more possibilities, is any of them sharper than the other? Which of these spaces gives the strongest result (if such a thing exists at all)? The answer is usually formulated in form of some *Sobolev inequality* or *Sobolev embedding.*

Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, having a Lipschitz boundary. We recall that given $p \in [1, \infty]$, the *Lebesgue space* $L^p(\Omega)$ is defined as the set of all Lebesgue measurable real-valued functions u on Ω such that its *Lebesgue norm* $\|u\|_{L^p(\Omega)}$ (or $\|u\|_p$ for short) is finite, where

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in \Omega} |u(x)| & \text{if } p = \infty. \end{cases}$$

The most classical form of the *Sobolev inequality* concerns Lebesgue spaces. It asserts that, given $1 < p < n$ and setting $p^* = \frac{np}{n-p}$, there exists $C > 0$ such that

$$\left(\int_\Omega |u(x)|^{p^*} \, dx\right)^{\frac{1}{p^*}} \leq C \left(\int_\Omega |(\nabla u)(x)|^p + |u(x)|^p \, dx\right)^{\frac{1}{p}}$$

for every weakly-differentiable function u for which the right hand side is finite. (Here and throughout, C denotes a constant independent of important quantities, not necessarily the same at each occurrence.)

It is useful to restate the Sobolev inequality in the form of a certain specific relation between function spaces. Given two normed linear spaces X and Y which both are subsets of the set of all Lebesgue-measurable real-valued functions defined on Ω, we say that X is (*continuously*) *embedded* into Y if $X \subset Y$ in the set-theoretical sense and, moreover, the identity operator is continuous from X into Y, that is, there exists a positive constant C such that for every $u \in X$ one has $\|u\|_Y \leq C\|u\|_X$. We shall write $X \hookrightarrow Y$ to denote that X is embedded into Y.

Given $p \in [1, \infty)$, we define the *Sobolev space* $W^{1,p}(\Omega)$ as the collection of all weakly-differentiable real-valued functions u on Ω such that $u \in L^p(\Omega)$ and $|\nabla u| \in L^p(\Omega)$, where ∇u is the weak gradient of u. It can be shown that the set $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)},$$

is a Banach space. We also introduce the Sobolev space of functions vanishing at the boundary of Ω, namely $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ (the functions having derivatives of all orders and a compact support in Ω) in $W^{1,p}(\Omega)$.

The above Sobolev inequality thus translates to an embedding between function spaces, namely

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad 1 < p < n. \tag{2.1}$$

The relation (2.1) is called the *Sobolev embedding.* The space $L^p(\Omega)$ will be considered as the *domain* space, while the space $L^{p^*}(\Omega)$ will be considered as the *range* (or *target*) space in the Sobolev embedding. We shall also say that the Sobolev space $W^{1,p}(\Omega)$ is *built upon* the Lebesgue space $L^p(\Omega)$.

The main object of our study will be the question of how *sharp* (or *optimal*) are the domain space and the range space in the Sobolev embedding. By calling a certain space "sharp" in certain relation we mean that this space cannot be replaced by another space of the same kind that would provide an essentially stronger result without violating the embedding.

In order to be able to decide whether an inequality involving a different function space is stronger or weaker then the given one, we first need to understand all the embedding relations between the function spaces which come to the picture. In the case of Lebesgue spaces this fact depends on the measure of the underlying domain Ω. If Ω is of finite measure and $1 \leq p \leq q \leq \infty$, then $L^q(\Omega) \hookrightarrow L^p(\Omega)$, and, moreover, this inclusion is strict when $p < q$.

It is important to observe that the answer to the optimality question heavily depends on the environment within which the optimality is considered. For example, the embedding (2.1) can not be essentially improved within the environment (or the category $\mathfrak{M}$ which appears in Definition 4.3) of Lebesgue spaces. This should be understood as follows. If we replace the domain Lebesgue space $L^p(\Omega)$ in (2.1) by an *essentially larger one*, that is, $L^q(\Omega)$ with $q < p$, then the resulting embedding

$$W^{1,q}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

is no longer true, since one can easily construct a function $u \notin L^{p^*}(\Omega)$ whose gradient lies in $L^q(\Omega)$. Put another way, if this embedding holds, then, necessarily, $p \leq q$ (which corresponds to Definition 4.3). Likewise, if we replace the *range* space $L^{p^*}(\Omega)$ by an *essentially smaller one*, that is, $L^r(\Omega)$ with $r > p^*$, then, again, the resulting embedding

$$W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$$

does not hold. In this sense, the embedding (2.1) is, at least within the environment of Lebesgue spaces, *sharp*, and it cannot be effectively improved.

However, optimality of function spaces, considered within such a special class as that of Lebesgue spaces, is not always entirely satisfactory. The Lebesgue scale is simply not delicate enough in order to describe all the interesting details about embeddings. This fact is perhaps best illustrated by the so-called *limiting* or *critical* case of the Sobolev embedding (2.1), corresponding to the case $p = n$. We note that

$$\lim_{p \to n_-} p^* = \infty.$$

Based on this observation, one might be tempted to believe that

$$W^{1,n}(\Omega) \hookrightarrow L^\infty(\Omega).$$

However, this embedding is not true. Again, it is not difficult to construct an essentially unbounded function with an appropriate power-logarithmic singularity which lies in $W^{1,n}(\Omega)$. We conclude that unless we are willing to abandon the environment of Lebesgue spaces, the only information which we can extract for the limiting embedding is

$$W^{1,n}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for every } q < \infty. \tag{2.2}$$

There is now again no room for improvement within the Lebesgue scale, but one would be very reluctant to call this result "optimal". It is in fact quite unsatisfactory because it does not provide us with any definite range function space. (One might be tempted to take intersection of all the spaces $L^q(\Omega)$ but it would be difficult to furnish it with a reasonable norm.) What we have to do is to look for more general function spaces which would describe this case in a more manageable way.

Our first refinement of Lebesgue spaces involves the class of the so-called *Orlicz spaces*. Assume that $A : [0, \infty) \to [0, \infty)$ is a convex strictly increasing function satisfying

$$\lim_{t \to 0+} \frac{A(t)}{t} = 0, \quad \text{and} \quad \lim_{t \to \infty} \frac{A(t)}{t} = \infty.$$

Then we say that A is a *Young function*. The collection $L^A(\Omega)$ of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\|u\|_{L^A(\Omega)} < \infty$, where

$$\|u\|_{L^A(\Omega)} = \inf \left\{ \lambda > 0; \int_\Omega A\left(\frac{|u(x)|}{\lambda}\right) dx \le 1 \right\},$$

is called the *Orlicz space*. The assumptions on A guarantee that the functional $\|u\|_{L^A(\Omega)}$ is a norm with respect to which the Orlicz space $L^A(\Omega)$ is a Banach space. The norm $\|u\|_{L^A(\Omega)}$ is called the *Luxemburg norm*.

It can be easily verified that if $A(t) = t^p$, then $L^A(\Omega) = L^p(\Omega)$, and therefore Orlicz spaces constitute a broader class than Lebesgue spaces. An important example of an Orlicz space which is not a Lebesgue space is the *Zygmund class*. Given $p \in (1, \infty)$ and $\alpha \in \mathbb{R}$ or $p = 1$ and $\alpha > 0$, then the function

$$A(t) = t^p \left(\log(1 + t)\right)^\alpha, \quad t \in (0, \infty),$$

is a Young function. We shall call the corresponding Orlicz space a *logarithmic Zygmund class* and denote it by $L^p(\log L)^\alpha(\Omega)$. Similarly, given $\alpha \ge 1$, the function

$$A(t) = \begin{cases} \exp\left(t^\alpha\right) - 1 & \text{if } \alpha > 1, \\ \exp(t) - t - 1 & \text{if } \alpha = 1, \end{cases} \quad t \in (0, \infty),$$

is a Young function. We shall call the corresponding Orlicz space an *exponential Zygmund class* and denote it by $\exp L^\alpha(\Omega)$.

One has of course some embedding relations between Orlicz spaces. Given two Young functions A and B, the embedding

$$L^B(\Omega) \hookrightarrow L^A(\Omega)$$

holds if and only if there exists a positive constant C such that, for every $t \in (0, 1)$, the inequality

$$A(t) \le B(Ct)$$

holds. It is however not difficult to construct two Orlicz spaces which are incomparable with respect to the embedding relation.

Equipped with Zygmund classes and (at least) the partial ordering of Orlicz spaces, we can formulate the following limiting case of the Sobolev embedding:

$$W^{1,n}(\Omega) \hookrightarrow \exp L^{n'}(\Omega), \tag{2.3}$$

where

$$n' = \frac{n}{n-1}.$$

This result is traditionally attributed to Trudinger [113]. However, it had appeared earlier, at least in certain modified form, for example in works of Yudovich [116], Peetre [93] and Pokhozhayev [100].

We can now reconsider our central question of how optimal this embedding is. It turns out, that, remarkably, the range space $\exp L^{n'}(\Omega)$ is sharp within the environment of Orlicz spaces. That is, $\exp L^{n'}(\Omega)$ is essentially *the smallest possible Orlicz space* that still renders this embedding true. In other words, if we replace the Zygmund class $\exp L^{n'}(\Omega)$ by any essentially smaller Orlicz space $L^A(\Omega)$, then the resulting embedding

$$W^{1,n}(\Omega) \hookrightarrow L^A(\Omega)$$

would no longer hold. This result is due to Hempel, Morris and Trudinger [61].

Now, the embedding (2.3) constitutes a substantial improvement of (2.2) but at the same time it opens new questions. For example, one can return back to (2.1) and ask whether the range space $L^{p^*}(\Omega)$ in that embedding is optimal within the context of *Orlicz* spaces. This question is legitimate because $L^{p^*}(\Omega)$ is a particular instance of an Orlicz space. We already know that this space is optimal range as a *Lebesgue* space, but now we are asking about its optimality in a fairly broader sense, so the question is sensible.

The answer turns out to be positive, as follows from the result of Cianchi [33].

A natural question is if Orlicz spaces have all the answers or whether there is some further improvement available. It turns out that the latter is true. We can still considerably improve both (2.1) and (2.3), but first we have to introduce a qualitatively new type of function spaces.

We denote by $\mathcal{M}(\Omega)$ the class of real-valued measurable functions on Ω and by $\mathcal{M}_+(\Omega)$ the class of non-negative functions in $\mathcal{M}(\Omega)$. Given $f \in \mathcal{M}(\Omega)$, its *non-increasing rearrangement* is defined by

$$f^*(t) = \inf \{\lambda > 0;\ |\{x \in \Omega;\ |f(x)| > \lambda\}| \le t\}, \quad t \in [0, \infty).$$

Then f^* is a non-negative non-increasing right-continuous function on $[0, \infty)$ with the important property that the measure of level sets of f equals that of f^*, that is, for every $\lambda > 0$, one has

$$|\{x \in \Omega;\ |f(x)| > \lambda\}| = |\{t \in [0, \infty);\ |f^*(t)| > \lambda\}|$$

(here $|\cdot|$ stands for both one-dimensional and higher-dimensional Lebesgue measure). An important consequence of this fact is the identity

$$\|f\|_{L^p(\Omega)} = \|f^*\|_{L^p(0,1)},$$

where $p \in [1, \infty]$, and, more generally,

$$\|f\|_{L^A(\Omega)} = \|f^*\|_{L^A(0,1)},$$

where A is an arbitrary Young function and the Orlicz space is defined for functions acting on $(0, \infty)$ in an obvious way.

It will be handy to introduce the *maximal non-increasing rearrangement* of f by

$$f^{**}(t) = t^{-1} \int_0^t f^*(s)\, ds, \quad t \in (0, \infty).$$

Then f^{**} is a non-negative non-increasing continuous function on $(0, \infty)$ satisfying $f^{**}(t) \ge f^*(t)$ for every $f \in \mathcal{M}(0, 1)$ and $t \in (0, \infty)$. It has the remarkable property that the operation $f \mapsto f^{**}$ is subadditive in the sense that

$$(f+g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \quad f, g \in \mathcal{M}(0,1),\ t \in (0,\infty).$$

It should be noted that the operation $f \mapsto f^*$ is *not* subadditive. Instead, one only has

$$(f+g)^*(s+t) \leq f^*(s) + g^*(t), \quad f, g \in \mathcal{M}(0,1),\ s, t \in (0,\infty),$$

which is often used in the particular form

$$(f+g)^*(t) \leq f^*(\tfrac{t}{2}) + g^*(\tfrac{t}{2}), \quad f, g \in \mathcal{M}(0,1),\ t \in (0,\infty).$$

Equipped with the operation $f \mapsto f^*$, we shall introduce new important classes of function spaces. We start with the so-called *two-parameter Lorentz spaces*.

Assume that $0 < p, q \leq \infty$. Then the *Lorentz space* $L^{p,q}(\Omega)$ is the collection of all $f \in \mathcal{M}(\Omega)$ such that $\|f\|_{L^{p,q}(\Omega)} < \infty$, where

$$\|f\|_{L^{p,q}(\Omega)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L^q(0,1)}.$$

The functional $\|\cdot\|_{L^{p,q}(\Omega)}$ is not always an r.i. norm. It follows from a more general result of Lorentz ([80]) that it is an r.i. norm if and only if $1 \leq q \leq p \leq \infty$. More generally, it is *equivalent* to an r.i. norm if and only if one of the conditions

$$\begin{cases} 1 < p < \infty,\ 1 \leq q \leq \infty, \\ p = q = 1, \\ p = q = \infty, \end{cases} \tag{2.4}$$

is satisfied. Lorentz spaces have an important *nesting property* with respect to the second parameter. Indeed, for every $p \in (0,\infty]$ and $0 < q < r \leq \infty$, we have

$$L^{p,q}(\Omega) \hookrightarrow L^{p,r}(\Omega), \tag{2.5}$$

and this embedding is strict. We also define the space $L^{(p,q)}(\Omega)$ as the collection of all $f \in \mathcal{M}(\Omega)$ such that $\|f\|_{L^{(p,q)}(\Omega)} < \infty$, where

$$\|f\|_{L^{(p,q)}(\Omega)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}(t)\|_{L^q(0,1)}.$$

Since $f^{**}(t) \geq f^*(t)$ for every $f \in \mathcal{M}(\Omega)$ and every $t \in (0,1)$, we always have

$$L^{(p,q)}(\Omega) \hookrightarrow L^{p,q}(\Omega).$$

It can be proved using an appropriate version of the Hardy inequality, that, in fact,

$$L^{(p,q)}(\Omega) = L^{p,q}(\Omega) \quad \text{whenever } 1 < p \leq \infty.$$

In the case when $p = 1$, the situation is different. More precisely, for every $q \in [1,\infty]$, the inclusion

$$L^{(1,q)}(\Omega) \hookrightarrow L^{1,q}(\Omega)$$

is strict.

With the help of Lorentz spaces, we can essentially refine the Sobolev embedding (2.1) and obtain

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*,p}(\Omega), \quad 1 < p < n. \tag{2.6}$$

Thanks to (2.5) and the obvious inequality $p < p^*$, this is indeed a non-trivial improvement of the range space in (2.1). The embedding (2.6) is due to Peetre [93], but it can be also traced in works of O'Neil [88] and Hunt [63].

A natural question now arises, whether a similar refinement involving the non-increasing rearrangement of a function is possible also for the limiting embedding (2.3). The answer is positive again, but in order to carry out the details we need to introduce the so-called *Lorentz–Zygmund spaces which constitute* a function scale which refines both Lorentz spaces and Zygmund classes. Its overlap with Orlicz spaces is non-trivial.

Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. Then the *Lorentz–Zygmund space* $L^{p,q;\alpha}(\Omega)$ is the collection of all $f \in \mathcal{M}(\Omega)$ such that $\|f\|_{L^{p,q;\alpha}(\Omega)} < \infty$, where

$$\|f\|_{L^{p,q;\alpha}(\Omega)} := \|t^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\tfrac{e}{t}\right) f^*(t)\|_{L^q(0,1)}.$$

Occasionally we will have to work with a modification of Lorentz–Zygmund spaces in which f^* is replaced by f^{**}. We denote such space by $L^{(p,q;\alpha)}(\Omega)$, hence

$$\|f\|_{L^{(p,q;\alpha)}(\Omega)} := \|t^{\frac{1}{p}-\frac{1}{q}} \log^{\alpha}\left(\tfrac{e}{t}\right) f^{**}(t)\|_{L^q(0,1)}.$$

These spaces were introduced and studied by Bennett and Rudnick [14]. If one of the following conditions

$$\begin{cases} 1 < p < \infty,\ 1 \leq q \leq \infty,\ \alpha \in \mathbb{R}; \\ p = 1,\ q = 1,\ \alpha \geq 0; \\ p = \infty,\ q = \infty,\ \alpha \leq 0; \\ p = \infty,\ 1 \leq q < \infty,\ \alpha + \frac{1}{q} < 0, \end{cases} \tag{2.7}$$

is satisfied, then $L^{p,q;\alpha}(0,1)$ is a rearrangement-invariant Banach function space. Assume that one of the conditions in (2.7) is satisfied. Then the associate space $(L^{p,q;\alpha})'(\Omega,\nu)$ of the Lorentz–Zygmund space $L^{p,q;\alpha}(\Omega,\nu)$ satisfies (up to equivalent norms)

$$\left(L^{p,q;\alpha}\right)'(\Omega,\nu) = \begin{cases} L^{p',q';-\alpha}(\Omega,\nu) & \text{if} \quad 1 < p < \infty,\ 1 \leq q \leq \infty,\ \alpha \in \mathbb{R}; \\ L^{\infty,\infty;-\alpha}(\Omega,\nu) & \text{if} \quad p = 1,\ q = 1,\ \alpha \geq 0; \\ L^{1,1;-\alpha}(\Omega,\nu) & \text{if} \quad p = \infty,\ q = \infty,\ \alpha \leq 0; \\ L^{(1,q';-\alpha-1)}(\Omega,\nu) & \text{if} \quad p = \infty,\ 1 \leq q < \infty,\ \alpha + \frac{1}{q} < 0 \end{cases} \tag{2.8}$$

(see [92, Theorems 6.11 and 6.12]). Moreover,

$$L^{(p,q;\alpha)}(\Omega,\nu) = \begin{cases} L^{p,q;\alpha}(\Omega,\nu) & \text{if } 1 < p \leq \infty; \\ L^{1,1;\alpha+1}(\Omega,\nu) & \text{if } p = q = 1,\ \alpha \geq 0, \end{cases} \tag{2.9}$$

and

$$L^p(\Omega,\nu) \hookrightarrow L^{(1,q)}(\Omega,\nu) \quad \text{for every } 1 < p \leq \infty,\ 1 \leq q \leq \infty \tag{2.10}$$

(se [92, Theorem 3.16 (i),(ii)]). The embedding

$$L^{p,q;\alpha}(\Omega) \hookrightarrow L^{p,r;\beta}(\Omega)$$

holds if and only if one of the conditions

$$\begin{cases} 0 < p < \infty,\ 0 < q \leq r \leq \infty,\ \alpha \geq \beta; \\ 0 < p \leq \infty,\ 0 < r < q \leq \infty,\ \alpha + \frac{1}{q} > \beta + \frac{1}{r}; \\ p = \infty,\ 0 < q \leq r \leq \infty,\ \alpha + \frac{1}{q} \geq \beta + \frac{1}{r} \end{cases} \tag{2.11}$$

is satisfied.

The Lorentz–Zygmund spaces include as particular examples Lebesgue spaces, two-parameter Lorentz spaces and Zygmund classes of both types. Indeed, we have

$$L^{p,q;\alpha}(\Omega) = L^{p,q}(\Omega) \quad \text{when } \alpha = 0 \text{ and } 0 < p, q \le \infty, \tag{2.12}$$

$$L^{p,p;\alpha}(\Omega) = L^{p}(\log L)^{p\alpha}(\Omega) \quad \text{when } p \in [1,\infty) \text{ and } \alpha > 0, \tag{2.13}$$

$$L^{\infty,\infty;-\frac{1}{\alpha}}(\Omega) = \exp L^{\alpha}(\Omega) \quad \text{when } \alpha > 0. \tag{2.14}$$

Equipped with Lorentz–Zygmund spaces, we have the following refinement of (2.3):

$$W^{1,n}(\Omega) \hookrightarrow L^{\infty,n;-1}(\Omega). \tag{2.15}$$

The first one to note this fact was Maz'ya who formulated it in a somewhat implicit form involving capacitary estimates, see [83, pp. 105 and 109]. Explicit formulations were given by Hansson in [62] and Brézis–Wainger in [22], the result can be also traced in the work of Brudnyi [24]. A more general assertion was later proved by Cwikel and Pustylnik [48].

The range space $L^{\infty,n;-1}(\Omega)$ in (2.15) is a very interesting function space. It is a Lorentz–Zygmund space but it is neither a Zygmund class nor an Orlicz spaces. It follows from the embeddings in (2.11) and the identity (2.14) that

$$L^{\infty,n;-1}(\Omega) \hookrightarrow \exp L^{n'}(\Omega),$$

and that this inclusion is strict. Consequently, (2.15) is an essential improvement of (2.3).

The embedding (2.15) can be viewed in some sense as the limiting case of (2.6) when $p \to n_+$. Indeed, both these results allow a unified approach, as shown in [81], where it was noticed that, for $1 < p < n$, we have

$$\int_0^1 t^{\frac{p}{p^*}-1} u^*(t)^p \, dt \le C \int_\Omega |\nabla u(x)|^p \, dx$$

for all $u \in W_0^{1,p}(\Omega)$, while, in the limiting case, we have

$$\int_0^1 \left(\frac{u^*(t)}{\log\left(\frac{e}{t}\right)} \right)^n \frac{dt}{t} \le C \int_\Omega |\nabla u|^n(x) \, dx$$

for all $u \in W_0^{1,n}(\Omega)$. Both these results were proved in an elementary way by first establishing a weak version of the Sobolev–Gagliardo–Nirenberg embedding, namely

$$\lambda \left(|\{|u| \ge \lambda\}| \right)^{\frac{1}{n'}} \le C \int_\Omega |\nabla u| \, dx, \quad u \in W_0^{1,1}(\Omega), \ \lambda > 0,$$

and then using a truncation argument due to Maz'ya (see [84]). The results of [81] show that yet further improvement of (2.15) is possible. Namely, one has

$$W_0^{1,n}(\Omega) \hookrightarrow W_n(\Omega),$$

where, for $0 < p \le \infty$, the space $W_p(\Omega)$ is defined as the family of all measurable functions on Ω for which

$$\|u\|_{W_p(\Omega)} = \begin{cases} \left(\int_0^1 \left(u^*(\frac{t}{2}) - u^*(t) \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty & \text{when } p < \infty; \\ \sup\limits_{0<t<1} \left(u^*(\frac{t}{2}) - u^*(t) \right) & \text{when } p = \infty. \end{cases}$$

The space $W_p(\Omega)$ has some interesting properties. For example:

(i) $\|\chi_E\|_{W_p(\Omega)} = (\log 2)^{\frac{1}{p}}$ for every measurable $E \subset \Omega$ and $p \in (0, \infty)$;
(ii) $W_1(\Omega) = L^\infty(\Omega)$;
(iii) for $p \in [1, \infty)$, each integer-valued $u \in W_p(\Omega)$ is bounded;
(iv) for $p \in (1, \infty)$, $W_p(\Omega)$ is not a linear set;
(v) for $p \in (1, \infty)$, $W_p(\Omega) \hookrightarrow L^{\infty,p;-1}(\Omega)$, and this embedding is strict;
(vi) $W_p(\Omega) \hookrightarrow W_q(\Omega)$ for every $0 < p < q \le \infty$, and this embedding is strict.

The norm of the space $W_p(\Omega)$ involves the functional $f^*(\frac{t}{2}) - f^*(t)$. In [11], Bastero, Milman and Ruiz showed that it can be equivalently replaced with $f^{**}(t) - f^*(t)$. The quantity $f^{**}(t) - f^*(t)$, which measures, in some sense, the oscillation of f, had been used in function spaces theory before. Function spaces involving this functional have been particularly popular since 1981 when Bennett, De Vore and Sharpley [13] introduced the function space "weak L^∞", the rearrangement-invariant space of functions for which $f^{**}(t) - f^*(t)$ is bounded.

The problem of optimality of function spaces in Sobolev embeddings can be also viewed from a reversed angle. So far we have focused solely on the question of optimality of the range space in various contexts. However, one can also ask whether the *domain* space is optimal. For example, it is clear that (2.1) and (2.6) have the best possible Lebesgue domain spaces. We can however ask whether these domain spaces are also optimal as *Orlicz* spaces.

The answer is interesting and perhaps even surprising. In the non-limiting embedding (2.1), the space $L^p(\Omega)$ is indeed the optimal Orlicz range for $L^{p^*}(\Omega)$ ([96, Corollary 4.9]). On the other hand, the situation in the limiting case is quite different. First of all, the answer is negative, that is, the space $L^n(\Omega)$ is *not* the largest Orlicz space for which the inequality (2.3) holds, since one can explicitly construct an essentially bigger such Orlicz space. However, as a construction in [96, Theorem 4.5] shows, the situation is even more peculiar. Namely, not only that $L^n(\Omega)$ is not the largest Orlicz space in (2.3) holds, but, oddly enough, *there is no such optimal Orlicz space at all.* This should be understood as follows. Suppose that $L^A(\Omega)$ is an Orlicz space such that

$$W^1 L^A(\Omega) \hookrightarrow \exp L^{n'}(\Omega).$$

Then one can explicitly construct a Young function B such that the corresponding Orlicz space $L^B(\Omega)$ is essentially larger than $L^A(\Omega)$, but still the embedding

$$W^1 L^B(\Omega) \hookrightarrow \exp L^{n'}(\Omega)$$

holds. In the next iteration of this process, to this $L^B(\Omega)$, we can construct a yet bigger such space, and so on. In a way, this result resembles the unsatisfactory situation which we have seen before and in which we find an entire scale of

Lebesgue spaces at the range position in (2.2). Here, again, one has something like an 'open set of function spaces' with no definite endpoint. We conclude that not even the (apparently rather fine) class of Orlicz spaces is delicate enough to provide all satisfactory answers.

In an analogous manner we can ask about the optimality of the domain function space $L^n(\Omega)$ in (2.3). It was shown in [51] that, interestingly, one can deduce the following embedding:

$$W^1\left(L^{n,1;-\frac{1}{n'}} + L^{n,\infty;\frac{1}{n}}\right)(\Omega) \hookrightarrow \exp L^{n'}(\Omega).$$

Complemented with

$$L^n(\Omega) \hookrightarrow \left(L^{n,1;-\frac{1}{n'}} + L^{n,\infty;\frac{1}{n}}\right)(\Omega),$$

the inclusion being strict, this gives a rather unexpected non-trivial improvement of the domain space in (2.3), qualitatively different from the above-mentioned one, built on Orlicz spaces.

At this moment, the only sensible conclusion is that we have seen perhaps too many examples and that the situation is an unpleasant mess. The solution is in considering the situation within some reasonable common environment that would provide a roof for all, or, at least, most of the function spaces mentioned so far rather than grappling with the intrinsic difficulties of each of the function classes separately. The appropriate such environment is that of the so-called *rearrangement-invariant spaces.*

3. Rearrangement-invariant Banach function spaces

We say that a functional $\|\cdot\|_{X(0,1)} : \mathcal{M}_+(0,1) \to [0,\infty]$ is a *function norm*, if, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}_+(0,1)$, and every $\lambda \geq 0$, the following properties hold:

(P1) $\|f\|_{X(0,1)} = 0$ if and only if $f = 0$; $\|\lambda f\|_{X(0,1)} = \lambda\|f\|_{X(0,1)}$; $\|f+g\|_{X(0,1)} \leq \|f\|_{X(0,1)} + \|g\|_{X(0,1)}$;

(P2) $f \leq g$ a.e. implies $\|f\|_{X(0,1)} \leq \|g\|_{X(0,1)}$;

(P3) $f_j \nearrow f$ a.e. implies $\|f_j\|_{X(0,1)} \nearrow \|f\|_{X(0,1)}$;

(P4) $\|1\|_{X(0,1)} < \infty$;

(P5) $\int_0^1 f(x)\,dx \leq C\|f\|_{X(0,1)}$ for some constant C independent of f.

If, in addition,

(P6) $\|f\|_{X(0,1)} = \|g\|_{X(0,1)}$ whenever $f^* = g^*$,

we say that $\|\cdot\|_{X(0,1)}$ is a *rearrangement-invariant function norm.*

With any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, it is associated another functional on $\mathcal{M}_+(0,1)$, denoted by $\|\cdot\|_{X'(0,1)}$, and defined, for $g \in \mathcal{M}_+(0,1)$, as

$$\|g\|_{X'(0,1)} = \sup_{\substack{f\geq 0 \\ \|f\|_{X(0,1)}\leq 1}} \int_0^1 f(s)g(s)\,ds.$$

It turns out that $\|\cdot\|_{X'(0,1)}$ is also a rearrangement-invariant function norm, which is called the *associate function norm* of $\|\cdot\|_{X(0,1)}$.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, the *rearrangement-invariant space* (*r.i. space* for short) $X(\Omega,\nu)$ is defined as the collection of all functions $u \in \mathcal{M}(\Omega,\nu)$ such that the expression

$$\|u\|_{X(\Omega,\nu)} = \|u^*\|_{X(0,1)} \tag{3.1}$$

is finite. Such expression defines a norm on $X(\Omega,\nu)$, and the latter is a Banach space endowed with this norm, called a rearrangement-invariant space. Moreover, $X(\Omega,\nu) \subset \mathcal{M}_0(\Omega,\nu)$ for any rearrangement-invariant space $X(\Omega,\nu)$. The space $X(0,1)$ is called the *representation space* of $X(\Omega,\nu)$.

A basic tool for working with rearrangement-invariant spaces is the *Hardy–Littlewood–Pólya* (HLP) *principle* (see e.g. [15, Chapter 2, Theorem 4.6]). It asserts that $f^{**}(t) \leq g^{**}(t)$ for every $t \in (0,1)$ implies $\|f\|_{X(\Omega)} \leq \|g\|_{X(\Omega)}$ for every r.i. space $X(\Omega)$.

The *inequality of Hardy and Littlewood* states that

$$\int_\Omega |f(x)g(x)|\,dx \leq \int_0^1 f^*(t)g^*(t)\,dt, \quad f,g \in \mathcal{M}(\Omega). \tag{3.2}$$

Let $X(\Omega)$ be an r.i. space. Then, the function $\varphi_X : [0,1] \to [0,\infty)$ given by

$$\varphi_X(t) = \begin{cases} \|\chi_{(0,t)}\|_{X(0,1)}, & \text{for } t \in (0,1], \\ 0 & \text{for } t = 0, \end{cases}$$

is called the *fundamental function* of $X(\Omega)$. For every r.i. space $X(\Omega)$, its fundamental function φ_X is *quasiconcave* on $[0,1]$, i.e., it is non-decreasing on $[0,1]$, $\varphi_X(0) = 0$, and $\frac{\varphi_X(t)}{t}$ is non-increasing on $(0,1]$. Moreover, one has

$$\varphi_X(t)\varphi_{X'}(t) = t, \quad \text{for } t \in [0,1].$$

Given an r.i. space $X(\Omega)$, we can define the *Marcinkiewicz space* $M_X(\Omega)$, corresponding to $X(\Omega)$, as the set of all $f \in \mathcal{M}(\Omega)$ such that

$$\|f\|_{M_X(\Omega)} := \sup_{t\in[0,1]} \varphi_X(t) f^{**}(t) < \infty.$$

Then, $M_X(\Omega)$ is, again, an r.i. space whose fundamental function is φ_X, and it is the largest such r.i. space. In particular, when $Z(\Omega)$ is any other r.i. space whose fundamental function is also φ_X, then, necessarily

$$Z(\Omega) \hookrightarrow M_X(\Omega).$$

If X is an r.i. space and $f \in X$, we say that f has an *absolutely continuous norm* in X if

$$\|f\chi_{E_n}\|_X \to 0$$

for every sequence $\{E_n\}_{n=1}^\infty$ satisfying $E_n \to \emptyset$ μ-a.e. We define

$$X_a = \{f \in X;\ f \text{ has an absolute continuous norm in } X\}.$$

If $X_a = X$, we say that the space X has an *absolutely continuous norm.*

For a comprehensive treatment of r.i. spaces we refer the reader to [15].

Assume that $m \in \mathbb{N}$. Given a rearrangement-invariant space $X(\Omega)$, we define the m-th order Sobolev space $W^m X(\Omega)$ as the collection of all m-times weakly differentiable functions u in Ω such that $|D^m u| \in X(\Omega)$. Our main goal is to compare the size of u with that of its *m-th gradient*, $|D^m u|$, in norms of two function spaces, preferably rearrangement invariant, where $D^m u = \left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)_{0\le|\alpha|\le m}$ and $|D^m u|$ is its Euclidean length. Thus, more precisely, we will be interested in determining those rearrangement-invariant spaces, $X(\Omega)$ and $Y(\Omega)$ for which one has

$$\|u\|_{Y(\Omega)} \le C \, \||D^m u|^*(t)\|_{X(0,1)}\,, \quad u \in W^m X(\Omega),$$

or, written as a Sobolev embedding,

$$W^m X(\Omega) \hookrightarrow Y(\Omega). \tag{4.1}$$

More specifically, we would like to know that $X(\Omega)$ and $Y(\Omega)$ are *optimal* in the sense that $X(\Omega)$ cannot be replaced by an essentially larger r.i. space and $Y(\Omega)$ cannot be replaced by an essentially smaller one.

The principal idea of our approach to embeddings can be formulated as follows. The first step is to reduce a Sobolev embedding to a one-dimensional inequality involving an integral operator. In the next step, one would hope to be able to use available knowledge about the action of such operators on function spaces. For the first-order embedding, this was done in [50].

Although the results in [50] are formulated only for Sobolev spaces of functions vanishing on the boundary of Ω, by the combination of the Stein extension theorem [4, Theorem 5.24] with an interpolation argument based on the De Vore–Scherer theorem [49] or [15, Chapter 5, Theorem 5.12, p. 360], they can be relatively easily extended to bounded domains with Lipschitz boundary. In this approach, the Sobolev space $W^m X(\Omega)$ is extended to $W^m X(\mathbb{R}^n)$ and then restricted again to $W^m X(\Omega_1)$ with $\Omega_1 \supset \Omega$. The details can be found in [67, proof of Theorem 4.1]. Yet more general results will be presented in Section 10.

The key result in [50] is the following *reduction theorem.*

Theorem 4.1. *Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. Then the Sobolev embedding*

$$W^1 X(\Omega) \hookrightarrow Y(\Omega)$$

holds if and only if there exists a constant $C > 0$ such that

$$\left\| \int_t^1 f(s) s^{\frac{1}{n}-1}\, ds \right\|_{Y(0,1)} \le C\, \|f\|_{X(0,1)}\,, \quad f \in \mathcal{M}_+(0,1).$$

This theorem concerns the first-order embeddings. The question how to obtain a higher-order version of the reduction theorem has been treated by several authors. While the ‘only if’ part is rather straightforward and easily adaptable, the proof of the ‘if’ part of Theorem 4.1 involves a version of the Pólya–Szegö inequality due to Talenti [111], whose higher-order version is unavailable without certain restrictions. In 2004, Cianchi [34] obtained the reduction theorem for the case $m = 2$ by overcoming certain considerable technical difficulties and using some special estimates for second-order derivatives. In [67], the reduction theorem was extended to every

m satisfying $1 \le m \le n-1$ by a method based on interpolation techniques and properties of Hardy-type operators involving suprema (see the operator $T_{\frac{n}{m}}$ treated below). Finally, the result involving every $m \in \mathbb{N}$ with no restrictions was obtained in [44]. It reads as follows.

Theorem 4.2. *Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. Then, the Sobolev embedding* (4.1) *holds if and only if*

$$\left\| \int_t^1 f(s) s^{\frac{m}{n}-1}\, ds \right\|_{Y(0,1)} \le C \, \|f\|_{X(0,1)}, \quad f \in \mathcal{M}_+(0,1).$$

We will now show how Theorem 4.2 can be used to characterize the largest rearrangement-invariant domain space and the smallest rearrangement-invariant range space in the Sobolev embedding (4.1). We start with a general definition of optimal partnership.

Definition 4.3. Let $\mathfrak{M}$ be some category of rearrangement-invariant spaces. Assume that $X(\Omega)$ and $Y(\Omega)$ are rearrangement-invariant spaces. We say that $Y(\Omega)$ is the *optimal range* partner for $X(\Omega)$ in the Sobolev embedding (4.1) within the category $\mathfrak{M}$ if the following three conditions are satisfied:

- $W^m X(\Omega) \hookrightarrow Y(\Omega)$;
- $Y(\Omega) \in \mathfrak{M}$;
- if $Z(\Omega) \in \mathfrak{M}$ satisfies $W^m X(\Omega) \hookrightarrow Z(\Omega)$, then $Y(\Omega) \hookrightarrow Z(\Omega)$.

We say that $X(\Omega)$ is the *optimal domain* partner for $Y(\Omega)$ in the Sobolev embedding (4.1) within the category $\mathfrak{M}$ if the following three conditions are satisfied:

- $W^m X(\Omega) \hookrightarrow Y(\Omega)$;
- $Y(\Omega) \in \mathfrak{M}$;
- if $Z(\Omega) \in \mathfrak{M}$ satisfies $W^m Z(\Omega) \hookrightarrow Y(\Omega)$, then $Z(\Omega) \hookrightarrow X(\Omega)$.

We say that $(X(\Omega), Y(\Omega))$ form an *optimal pair* in the Sobolev embedding if $Y(\Omega)$ is the optimal range partner for $X(\Omega)$ and at the same time $X(\Omega)$ is the optimal domain partner for $Y(\Omega)$ in (4.1) within $\mathfrak{M}$.

Two sublinear operators T and T' from $\mathcal{M}_+(0,1)$ into $\mathcal{M}_+(0,1)$ will be called *mutually associate* (with respect to the L^1 pairing), if

$$\int_0^1 Tf(s) g(s)\, ds = \int_0^1 f(s) T'g(s)\, ds \tag{4.2}$$

for every $f, g \in \mathcal{M}_+(0,1)$. For associate operators, the following general observation is rather useful.

Lemma 4.4. *Let T and T' be mutually associate operators, and let $X(0,1)$ and $Y(0,1)$ be rearrangement-invariant spaces. Then,*

$$T : X(0,1) \to Y(0,1) \quad \textit{if and only if} \quad T' : Y'(0,1) \to X'(0,1),$$

and

$$\|T\| = \|T'\|.$$

Proof. The conclusion is a consequence of the chain:

$$\begin{aligned}
\|T\| &= \sup_{\substack{f\geq 0\\ \|f\|_{X(0,1)}\leq 1}} \|Tf\|_{Y(0,1)} = \sup_{\substack{f\geq 0\\ \|f\|_{X(0,1)}\leq 1}} \sup_{\substack{g\geq 0\\ \|g\|_{Y'(0,1)}\leq 1}} \int_0^1 Tf(s)g(s)\,ds \\
&= \sup_{\substack{g\geq 0\\ \|g\|_{Y'(0,1)}\leq 1}} \sup_{\substack{f\geq 0\\ \|f\|_{X(0,1)}\leq 1}} \int_0^1 f(s)T'g(s)\,ds = \sup_{\substack{g\geq 0\\ \|g\|_{Y'(0,1)}\leq 1}} \|T'g\|_{X'(0,1)} \\
&= \|T'\|.
\end{aligned}$$

□

We now define the *weighted Hardy operator* $H_{\frac{n}{m}}$, given as

$$(H_{\frac{n}{m}}f)(t) := \int_t^1 f(s)s^{\frac{m}{n}-1}\,ds, \quad t\in(0,1),\ f\in\mathcal{M}_+(0,1),$$

and its associate operator

$$(H'_{\frac{n}{m}}f)(t) := t^{\frac{m}{n}-1}\int_0^t f(s)\,ds, \quad t\in(0,1),\ f\in\mathcal{M}_+(0,1).$$

Applying the operator $H_{\frac{n}{m}}$ to a *non-increasing* function g^*, we get

$$(H'_{\frac{n}{m}}g^*)(t) = t^{\frac{m}{n}}g^{**}(t) \quad t\in(0,1),\ g\in\mathcal{M}(\Omega).$$

It is important to observe that the functional

$$\left\|t^{\frac{m}{n}}g^{**}(t)\right\|_{X'(0,1)}, \quad g\in\mathcal{M}(\Omega),$$

is in fact an r.i. norm on $\mathcal{M}(\Omega)$. This is easy to verify as the only non-trivial part is the triangle inequality, which follows from the subadditivity of the operation $g\to g^{**}$. Note that Theorem 4.2 implies the following chain of equivalent statements:

$$\begin{aligned}
W^mX(\Omega)\hookrightarrow Y(\Omega) \quad &\Leftrightarrow \quad H_{\frac{n}{m}}: X(0,1)\to Y(0,1)\\
&\Leftrightarrow \quad H'_{\frac{n}{m}}: Y'(0,1)\to X'(0,1)\\
&\Leftrightarrow \quad \left\|t^{\frac{m}{n}}g^{**}(t)\right\|_{X'(0,1)} \leq C\left\|g\right\|_{Y'(\Omega)}, \quad g\in\mathcal{M}(\Omega).
\end{aligned}$$

The first equivalence is Theorem 4.2 and the second one is Lemma 4.4. The last equivalence is not entirely obvious. The implication "⇒" is just the restriction of the boundedness of the operator $H'_{\frac{n}{m}}$ to those functions on $(0,1)$ which are moreover non-increasing, so this implication is easy. The converse implication is subtler. It follows from the estimate $\int_0^t g(s)\,ds \leq \int_0^t g^*(s)\,ds$, which is a special case of the Hardy–Littlewood inequality (3.2). It is of interest to note that when we replace the operator $H'_{\frac{n}{m}}$ by $H_{\frac{n}{m}}$, then the corresponding equivalence would be no longer true. More precisely, the inequality

$$\left\|H_{\frac{n}{m}}g\right\|_{Y(0,1)} \leq C\|g\|_{X(0,1)}, \quad g\in\mathcal{M}(0,1),$$

implies

$$\left\|H_{\frac{n}{m}}g^*\right\|_{Y(0,1)} \leq C\|g\|_{X(0,1)}, \quad g\in\mathcal{M}(0,1),$$

but *not* vice versa. This illustrates that a Sobolev embedding is a rather delicate process that does not permit a direct duality.

All these ideas are summarized in the following theorem.

Theorem 4.5. *Let $X(\Omega)$ be an r.i. space. Let $Y(\Omega)$ be the r.i. space whose associate space, $Y'(\Omega)$, has the norm*

$$\|f\|_{Y'(\Omega)} := \|t^{\frac{m}{n}} f^{**}(t)\|_{X'(0,1)}, \quad f \in \mathcal{M}(\Omega).$$

Then, the Sobolev embedding (4.1) *holds, and $Y(\Omega)$ is the optimal* (*that is, the smallest possible*) *such r.i. space.*

Proof. Let $Z(\Omega)$ be any rearrangement-invariant function norm such that

$$W^m X(\Omega) \hookrightarrow Z(\Omega)$$

holds. Then, by Theorem 4.2, there exists a constant $C > 0$ such that

$$\left\| \int_t^1 f(s) s^{\frac{m}{n}-1}\, ds \right\|_{Z(0,1)} \leq C \|f\|_{X(0,1)}, \quad f \in \mathcal{M}_+(0,1).$$

In other words,

$$H_{\frac{n}{m}} : X(0,1) \to Z(0,1). \tag{4.3}$$

However, it is easy to observe that $Y(0,1)$ is the optimal rearrangement-invariant target space in (4.3), so we necessarily have

$$Y(0,1) \hookrightarrow Z(0,1).$$

This implies the optimality of the norm $\|\cdot\|_{Y(0,1)}$ in (4.1). $\square$

Theorem 4.5 constitutes an important and rather nice theoretical breakthrough in our search for optimal Sobolev embeddings. On the other hand, it is formulated in an implicit way and its application to a concrete example is not straightforward. In order to determine $Y(\Omega)$ for a given $X(\Omega)$, we have to be able to determine the associate space of the space normed by $g \mapsto \left\|t^{\frac{m}{n}} g^{**}(t)\right\|_{X'(0,1)}$. That might be quite difficult in practice. Even in the simplest possible instance when $X(\Omega) = L^p(\Omega)$, $1 \leq p < n$, this task is rather involved. Using a deep duality principle due to Sawyer ([103]), we obtain $Y(\Omega) = L^{p',p}$. Hence, in particular, we obtain that the range space in (2.6) is optimal within all rearrangement-invariant spaces. When $X(\Omega)$ is, for instance, an Orlicz space, the explicit evaluation of $Y(\Omega)$ becomes nearly impossible. Certain characterization, however, was found by Cianchi ([35]). In [50], the class of the so-called *Lorentz–Karamata spaces* was introduced, and the explicit formulas for the optimal range space were given in case when the domain space is one of these. The Lorentz–Karamata spaces are a generalization of Lorentz–Zygmund spaces which instead of logarithmic functions involve more general *slowly-varying functions.*

We will now apply Theorem 4.5 to a concrete example, a higher-order version of the Maz'ya–Hansson–Brézis–Wainger embedding (2.15).

Example 4.6. Let $X(\Omega) = L^{\frac{n}{m}}(\Omega)$. Then, by Theorem 4.5, its optimal range partner $Y(\Omega)$ is the associate space of $Y'(\Omega)$, determined by the norm

$$\|g\|_{Y'(\Omega)} = \|f^{**}(t)t^{\frac{m}{n}}\|_{L^{\frac{n}{n-m}}(0,1)}.$$

Now, by a duality principle of Sawyer [103], we obtain

$$Y(\Omega) = L^{\infty,\frac{n}{m};-1}(\Omega).$$

For $m = 1$, we recover (2.15), and we add a new information that this range space is the best possible among r.i. spaces. As mentioned above already, $W_n(\Omega)$ is still a slightly better range, but it is not an r.i. space for not being a linear set.

We shall now turn our attention to the question of characterizing the optimal *domain* function space assuming that the range space has been fixed. Likewise the characterization of the optimal range, this result that can be obtained as a consequence of the reduction theorem and it reads as follows.

Theorem 4.7. *Assume that $Y(\Omega)$ is an r.i. space such that $Y(\Omega) \hookrightarrow L^{\frac{n}{n-m},1}(\Omega)$. Then, the function space $X(\Omega)$, generated by the norm*

$$\|f\|_{X(\Omega)} = \sup_{h^*=f^*} \left\|H_{\frac{n}{m}}h\right\|_{Y(0,1)}, \quad f \in \mathcal{M}(\Omega),\ h \in \mathcal{M}(0,1), \tag{4.4}$$

is an r.i. space such that

$$H_{\frac{n}{m}} : X(0,1) \to Y(0,1)$$

(hence $W^mX(\Omega) \hookrightarrow Y(\Omega)$). Moreover, it is the optimal (largest) such space.

The requirement of the embedding of $Y(\Omega)$ into $L^{\frac{n}{n-m},1}(\Omega)$ is not restrictive as the space $L^{\frac{n}{n-m},1}(\Omega)$ is the range partner for the space $L^1(\Omega)$, the largest of all r.i. spaces. Therefore, larger spaces than $L^{\frac{n}{n-m},1}(\Omega)$ are not interesting range candidates.

Likewise Theorem 4.5, Theorem 4.7 is formulated in rather an implicit way, hence it can hardly be applied directly to a concrete example. Direct evaluation of $X(\Omega)$ from (4.4) is practically impossible. In the search of a simplification, several methods have been applied. Among functions in $\mathcal{M}(0,1)$ that are equimeasurable to a given function f in $\mathcal{M}(\Omega)$, there is one with an exceptional significance, namely f^* itself. So, a natural question is under what conditions one can replace in (4.4) $\sup_{h^*=f^*} \left\|H_{\frac{n}{m}}h\right\|_{Y(0,1)}$ by $\left\|H_{\frac{n}{m}}f^*\right\|_{Y(0,1)}$. Of course, only the inequality

$$\sup_{h^*=f^*} \left\|H_{\frac{n}{m}}h\right\|_{Y(0,1)} \le C \left\|H_{\frac{n}{m}}f^*\right\|_{Y(0,1)}$$

is in question, since the converse one is trivial. However (and this is the main problem here), the quantity on the right is not necessarily a norm. This shortcoming stems from the fact that the operation $f \mapsto f^*$ is not subadditive, as noted above, so the triangle inequality is not guaranteed. For example, for $Y(\Omega) = L^1(\Omega)$, it is easy to verify that $\left\|H_{\frac{n}{m}}f^*\right\|_{L^1(0,1)}$ is not a norm. In [50], a sufficient condition for $\left\|H_{\frac{n}{m}}f^*\right\|_{Y(0,1)}$ to be a norm was established, namely

$$\left\|H_{\frac{n}{m}}f^{**}\right\|_{Y(0,1)} \le C \left\|H_{\frac{n}{m}}f^*\right\|_{Y(0,1)}. \tag{4.5}$$

Replacing f^* with f^{**} immediately solves the triangle inequality problem, since the operation $f \mapsto f^{**}$ *is* subadditive. On the other hand the condition (4.5) is too strong for the practical use, because it rules out important limiting cases. For instance, it is easy to see that for the space $Y(\Omega) = L^{\frac{n}{n-m}}(\Omega)$, (4.5) is not true. In [95], another approach using special operators was elaborated. Finally, in [68], it was shown that a reasonable sufficient condition can be formulated in terms of the operator $T_{\frac{n}{m}}$, defined by

$$\left(T_{\frac{n}{m}} f\right)(t) := t^{-\frac{m}{n}} \sup_{t \le s < 1} s^{\frac{m}{n}} f^*(s), \quad f \in \mathcal{M}(0,1),\ t \in (0,1).$$

One readily shows that $T_{\frac{n}{m}}$ is bounded on $L^1(0,1)$ and also on the Lorentz space $L^{\frac{n}{m},\infty}(0,1)$. It turns out that the decisive role for the question of when $\left\|H_{\frac{n}{m}} f^*\right\|_{Y(0,1)}$ is a norm, is played by the result of the test whether or not $T_{\frac{n}{m}}$ is bounded on the associate space $Y'(0,1)$ of the representation space of $Y(\Omega)$. The above observation shows that, for example in the cases when $Y(\Omega) = L^\infty(\Omega)$ or $Y(\Omega) = L^{\frac{n}{n-m},1}(\Omega)$ this condition is satisfied. One can also show (either directly or by interpolation) that it is satisfied for every space $Y(\Omega) = L^{p,q}(\Omega)$ for $\frac{n}{n-m} < p < \infty$ and $1 \le q \le \infty$. Let us formulate the general result.

Theorem 4.8. *Let $Y(\Omega)$ be an r.i. space, satisfying*

$$T_{\frac{n}{m}} : Y'(0,1) \to Y'(0,1). \tag{4.6}$$

Then, given a rearrangement-invariant space $Y(\Omega)$, the space $X(\Omega)$, defined as the collection of all $f \in \mathcal{M}(\Omega)$ such that $\|f\|_{X(\Omega)} < \infty$, where

$$\|f\|_{X(\Omega)} = \left\| \int_t^1 f^*(s) s^{\frac{m}{n}-1}\, ds \right\|_{Y'(0,1)}, \quad f \in \mathcal{M}(\Omega),$$

is the optimal (largest) rearrangement-invariant space at the domain position in (4.1).

The condition (4.6) is reasonable and it preserves all the important limiting examples that had been ruled out by the condition (4.5). Our next aim is to demonstrate that this condition is in fact rather natural.

Example 4.9. We shall now consider the higher-order analogue of the Maz'ya–Hansson–Brézis–Wainger embedding (2.15). Assume that $X(\Omega) = L^{\frac{n}{m}}(\Omega)$. Then, by Example 4.6, the smallest rearrangement-invariant space $Y(\Omega)$ such that $W^1X(\Omega) \hookrightarrow Y(\Omega)$, is $L^{\infty,\frac{n}{m};-1}(\Omega)$. Our next aim is to show that $T_{\frac{n}{m}}$ is bounded on the associate space of $Y(0,1)$. As observed in Example 4.6, the associate space of $Y(0,1)$ is the space $L^{(1,\frac{n}{n-m})}(0,1)$. In order to prove the boundedness of $T_{\frac{n}{m}}$ on $L^{(1,\frac{n}{n-m})}(0,1)$, we first note that $T_{\frac{n}{m}}$ is bounded on $L^{1,\frac{n}{n-m}}(0,1)$, which is easier and which can be done either by a standard interpolation argument or using conditions for the weighted norm inequalities involving supremum operators from [37] or [55]. Next we show that $(T_{\frac{n}{m}} g)^{**}$ is comparable to $T_{\frac{n}{m}}(g^{**})$. Combining these two facts, we get the desired boundedness of $T_{\frac{n}{m}}$ on $L^{(1,\frac{n}{n-m})}(0,1)$, which is $Y'(0,1)$. Hence, according to Theorem 4.8, the optimal r.i. domain partner space, $\widetilde{X}(\Omega)$, has norm

$$\|g\|_{\widetilde{X}(\Omega)} = \|H_{\frac{n}{m}} g^*\|_{L^{\infty,\frac{n}{m};-1}(0,1)}.$$

Now, several interesting facts can be observed about this space. First, it indeed is strictly larger than $X(\Omega) = L^{\frac{n}{m}}(\Omega)$. In fact, it even has an essentially different fundamental function. Moreover, it is a qualitatively new type of a function space. In [96], several interesting properties of this space were established, for example its incomparability to several related known function spaces of Orlicz and Lorentz–Zygmund type.

5. Formulas for optimal spaces using the functional $f^{**} - f^{*}$

Let us summarize the questions we have studied so far. The "optimal range" problem requires us to find the optimal range partner to the given fixed space $X(\Omega)$. Let us denote this optimal partner by $Y_X(\Omega)$. In other words, the Sobolev embedding

$$W^m X(\Omega) \hookrightarrow Y_X(\Omega), \tag{5.1}$$

holds, and, moreover, $Y_X(\Omega)$ is the smallest possible such r.i. space. The "optimal domain" problem, perhaps somewhat a less frequent task in practice, but also of interest, is the converse one; given m and an r.i. space $Y(\Omega)$, find its optimal domain r.i. partner, let us call it $X_Y(\Omega)$, for $Y(\Omega)$, so that

$$W^m X_Y(\Omega) \hookrightarrow Y(\Omega),$$

holds and $X_Y(\Omega)$ is the largest possible such r.i. space.

At this stage, we have formulas for both $Y_X(\Omega)$ and $X_Y(\Omega)$, given by Theorems 4.5 and 4.7, respectively. As we have already noticed, these formulas are too implicit to allow for some practical use. Theorem 4.7 is particularly bad. In this section we will show that significant simplifications of these formulas, such as the one given by Theorem 4.8, are possible if we a-priori know that the given space has been chosen in such a way that it is an optimal partner for some other r.i. space.

We first need to introduce one more supremum operator. Let

$$\left(S_{\frac{n}{m}} f\right)(t) := t^{\frac{m}{n}-1} \sup_{0<s\le t} s^{1-\frac{m}{n}} f^*(s), \quad f \in \mathcal{M}(0,1),\ t \in (0,1).$$

Then, $S_{\frac{n}{m}}$ has the following endpoint mapping properties:

$$S_{\frac{n}{m}} : L^{\frac{n}{n-m},\infty}(0,1) \to L^{\frac{n}{n-m},\infty}(0,1) \quad \text{and} \quad S_{\frac{n}{m}} : L^{\infty}(0,1) \to L^{\infty}(0,1).$$

In what follows we shall write $A \approx B$ when there exists a positive constant independent of appropriate quantities such that

$$C^{-1} \le \frac{A}{B} \le C.$$

Our point of departure will be the following result from [68].

Theorem 5.1. *Let $X(\Omega)$ be an r.i. space, whose associate space satisfies $X'(\Omega) \hookrightarrow L^{\frac{n}{n-m},\infty}(\Omega)$. Then,*

$$\|f\|_{Y_X(\Omega)} \approx \sup_{\|S_{\frac{n}{m}} g\|_{X'(0,1)} \le 1} \int_0^1 t^{-\frac{m}{n}} \left(f^{**}(t) - f^*(t)\right) g^*(t)\, dt + \|f\|_{L^1(\Omega)}, \tag{5.2}$$

where $f \in \mathcal{M}(\Omega)$, $g \in \mathcal{M}_+(0,1)$.

The most innovative part of Theorem 5.1 is the new formula (5.2). The L^1-norm has just a cosmetic meaning, it's role is to guarantee that the resulting functional is a norm. The main term is formulated as some kind of duality involving the operator $S_{\frac{n}{m}}$. In cases when $S_{\frac{n}{m}}$ can be peeled off, the whole expression is considerably simpler.

Theorem 5.2. *An r.i. space $X(\Omega)$ is the optimal domain partner in* (4.1) *for some other r.i. space $Y(\Omega)$ if and only if*

$$S_{\frac{n}{m}} : X'(0,1) \to X'(0,1).$$

In that case,

$$\|f\|_{Y_X(\Omega)} \approx \left\|t^{-\frac{m}{n}}\left(f^{**}(t) - f^*(t)\right)\right\|_{X(0,1)} + \|f\|_{L^1(\Omega)}, \quad f \in \mathcal{M}(\Omega).$$

Again, an r.i. space $Y(\Omega)$ is the optimal range partner in (4.1) *for some other r.i. space $X(\Omega)$ if and only if*

$$T_{\frac{n}{m}} : Y'(0,1) \to Y'(0,1).$$

In that case,

$$\|f\|_{X_Y(\Omega)} \approx \left\|\int_t^1 f^*(s)s^{\frac{m}{n}-1}\,ds\right\|_{Y(0,1)}, \quad f \in \mathcal{M}(\Omega).$$

This result enables us to apply a new approach. We start with a given r.i. space $X(\Omega)$. We find the corresponding optimal range r.i. partner $Y_X(\Omega)$. Now, the embedding (5.1) has an optimal range, but it does not necessarily have an optimal domain, as Example 4.9 shows. We thus take one more step in order to get the optimal domain r.i. partner for $Y_X(\Omega)$, let us call it $\widetilde{X}(\Omega)$. In place of Theorem 4.7, let us now use the (somewhat more friendly) Theorem 4.8. This is legitimate because $Y_X(\Omega)$ is now known to be the optimal range partner for $X(\Omega)$, and Theorem 5.2 tells us that this is equivalent to the required boundedness of $T_{\frac{n}{m}}$ on ${Y_X}'(0,1)$. Altogether, we have

$$W^m X(\Omega) \subset W^m \widetilde{X}(\Omega) \hookrightarrow Y(\Omega).$$

Now we have two possibilities. Either $\widetilde{X}(\Omega)$ is equivalent to $X(\Omega)$ or it is an essentially larger space. It is an easy exercise to verify that further iterations of this process do not bring anything new, because after the two steps, the couple $(\widetilde{X}(\Omega), Y(\Omega))$ forms an optimal rearrangement-invariant pair in the Sobolev embedding.

The functional $f^{**}(t) - f^*(t)$, appearing in (5.2), should cause some natural concern. It is known ([28]) that function spaces whose norms involve this functional often do not enjoy nice properties such as linearity, lattice property or normability. For example, for $X(\Omega) = L^{\frac{n}{m}}(\Omega)$ (cf. [28, Remark 3.2]), all these properties for $Y_X(\Omega)$ are lost. It is instructive to compare this fact with Theorem 4.8, where this case is ruled out by the assumption $S_{\frac{n}{m}} : X'(0,1) \to X'(0,1)$. This makes the significance of the supremum operator more transparent; $S_{\frac{n}{m}}$ is bounded on $L^{\frac{n}{n-m},\infty}(0,1)$ but *not* on $L^{\frac{n}{n-m}}(0,1)$. This example is typical, and it illustrates the general principle: the boundedness of $S_{\frac{n}{m}}$ on $X'(0,1)$ guarantees that $Y_X(\Omega)$ is a norm.

Incidentally, certain care must be exercised always when the norm of a given function space depends on f^*. For illustration of this fact, see [47]. Let us just add that a detailed study of weighted function spaces based on the functional $f^{**} - f^*$ can be found in [28] and [29].

Theorem 5.2 now can be used to obtain a new description of the space $\widetilde{X}(\Omega)$.

Theorem 5.3. *Let $X(\Omega)$ be an r.i. space and let $\widetilde{X}(\Omega)$ be defined as above. Define the space $Z(\Omega)$ by*

$$\|g\|_{Z(\Omega)} := \|S_{\frac{n}{m}} g^{**}\|_{X'(0,1)}, \quad g \in \mathcal{M}(\Omega).$$

Then,

$$\widetilde{X}(\Omega) = Z'(\Omega).$$

The proofs of Theorems 5.2 and 5.3 reveal a very interesting link between optimality of r.i. spaces in Sobolev embeddings and their interpolation properties.

Given three Banach spaces $X(\Omega)$, $Y(\Omega)$, $Z(\Omega)$, such that all of them are subsets of $\mathcal{M}(\Omega)$, we say that $X(\Omega)$ is an *interpolation space* between $Y(\Omega)$ and $Z(\Omega)$, written

$$X(\Omega) \in \operatorname{Int}\left(Y(\Omega), Z(\Omega)\right),$$

if for every sublinear operator T which is bounded from $Y(\Omega)$ into itself and also from $Z(\Omega)$ into itself, is also bounded from $X(\Omega)$ into itself.

The connection between optimality and interpolation is obtained through the following theorem.

Theorem 5.4. *Let $X(\Omega)$ be an r.i. space. Then, the operator $T_{\frac{n}{m}}$ is bounded on $X'(0,1)$ if and only if $X(\Omega)$ is an interpolation space with respect to the pair $\left(L^{\frac{n}{n-m},1}(\Omega), L^\infty(\Omega)\right)$, a fact which we will denote as*

$$X(\Omega) \in \operatorname{Int}\left(L^{\frac{n}{n-m},1}(\Omega), L^\infty(\Omega)\right).$$

Similarly, the operator $S_{\frac{n}{m}}$ is bounded on $X(0,1)$ if and only if

$$X(\Omega) \in \operatorname{Int}\left(L^{\frac{n}{n-m},\infty}(\Omega), L^\infty(\Omega)\right).$$

In other words, r.i. spaces in a Sobolev embedding can be optimal (domains or range) partners for some other r.i. spaces if and only if they satisfy certain interpolation properties. Of course, for example, a very large space, which does not satisfy the interpolation property, can also be a range in a Sobolev embedding, but not the optimal one.

The formulas for optimal spaces given by Theorems 5.2 and 5.3 are still not as explicit as one would desire, but, at least, they show the problem in a new light. They also enable us to obtain explicit formulas for some examples such as Orlicz spaces, previously unavailable. We finish this section by an example that can be computed using Theorem 5.2.

Theorem 5.5. *Let A be a Young function for which there exists $r > 1$ with*

$$\tilde{A}(rt) \geq 2r^{\frac{n}{n-m}} \tilde{A}(t), \quad t \geq 1.$$

Then, the r.i. spaces $X(\Omega) = L^A(\Omega)$ and $Y(\Omega)$, whose norm is given by

$$\|f\|_{Y(\Omega)} := \left\|t^{-\frac{m}{n}} \left(f^{**}(t) - f^*(t)\right)\right\|_{L^A(0,1)} + \|f\|_{L^1(\Omega)}, \quad f \in \mathcal{M}(\Omega),$$

are optimal in (4.1).

6. Explicit formulas for optimal spaces in Sobolev embeddings

Our aim in this section is to establish explicit formulas for the spaces $Y_X(\Omega)$ and $\widetilde{X}(\Omega)$, given an r.i. space $X(\Omega)$. We recall that the formulas for these spaces which we have established so far, are expressed in terms of their associate spaces, namely,

$$\|f\|_{Y_X'(\Omega)} := \|f^{**}(t)t^{\frac{m}{n}}\|_{X'(0,1)}, \quad f \in \mathcal{M}(\Omega), \tag{6.1}$$

and

$$\|g\|_{\widetilde{X}'(\Omega)} := \|S_{\frac{n}{m}} g^{**}\|_{X'(0,1)}, \quad g \in \mathcal{M}(\Omega). \tag{6.2}$$

Our focus will be now on the problem how to get these constructions explicit. We first note that the expression for $Y_X'(\Omega)$ turns out to be unsatisfactory in that the function

$$t \to t^{\frac{m}{n}-1} \int_0^t g^*(s)\, ds, \quad t \in [0,\infty),\ g \in \mathcal{M}(\Omega),$$

need not be non-increasing. This complicates the construction of explicit formulas for $Y_X(\Omega)$. (However, see [50, Section 4] and [68, Section 4].) Our next theorem from [70] solves this difficulty.

Theorem 6.1. *Suppose $X(\Omega)$ is an r.i. space satisfying*

$$X(\Omega) \supset L^{\frac{n}{m},1}(\Omega).$$

Define the space $Z_X(\Omega)$ by

$$\|g\|_{Z_X(\Omega)} := \left\| t^{\frac{m}{n}-1} \int_0^t g^*(s) s^{-\frac{m}{n}}\, ds \right\|_{X'(0,1)}, \quad g \in \mathcal{M}(\Omega).$$

Then,

$$\|f\|_{Y_X(\Omega)} \approx \left\| t^{-\frac{m}{n}} f^*(t) \right\|_{Z_X'(0,1)}, \quad f \in \mathcal{M}(\Omega).$$

We note that this eliminates the above-mentioned problem, since the function

$$t \mapsto t^{\frac{m}{n}-1} \int_0^t g^*(s) s^{-\frac{m}{n}}\, ds, \quad t \in [0,\infty),$$

is non-increasing, being a constant multiple of a weighted average of a non-increasing function.

Theorem 6.1 is, again, rather involved. The proof uses delicate estimates and previously obtained optimality results for various integral and supremum operators. The remaining task is to compute associate spaces of $\widetilde{X}'(\Omega)$ and $Y_X'(\Omega)$. To this end, we use the Brudnyi–Kruglyak duality theory ([25]) and the interpolation methods using the k-functional, elaborated recently in [66].

The main result reads as follows.

Theorem 6.2. *Suppose $X(\Omega)$ is an r.i. space. Define the space $V_X(\Omega)$ by*

$$\|g\|_{V_X(\Omega)} := \|g^{**}(t^{1-\frac{m}{n}})\|_{X'(0,1)}, \quad g \in \mathcal{M}(\Omega).$$

Then,

$$\|g\|_{\widetilde{X}(\Omega)} \approx \big(k(t, g^*; L^1(0,1), L^{\frac{n}{m},1}(0,1))\big\|_{V_X'(0,1)}, \quad g \in \mathcal{M}(\Omega).$$

Moreover,

$$\|f\|_{Y_X(\Omega)} \approx \left\|k(t, s^{-\frac{m}{n}} f^*(s); L^1(0,1), L^{\frac{n}{m},\infty}(0,1))\right\|_{V'_X(0,1)}, \quad f \in \mathcal{M}(\Omega).$$

Let us now briefly indicate how the interpolation K-method comes in. Let X_1 and X_2 be Banach spaces, *compatible* in the sense that they are embedded in a common Hausdorff topological vector space H. Suppose $x \in X_1 + X_2$ and $t \in [0,\infty)$. The *Peetre K-functional* is defined by

$$K(t, x; X_1, X_2) := \inf_{x = x_1 + x_2} \left(\|x_1\|_{X_1} + t\|x_2\|_{X_2}\right), \quad t > 0.$$

It is an increasing concave function of t on $[0,\infty)$, so that the function

$$k\left(t, x; X_1, X_2\right) := \frac{d}{dt} K\left(t, x; X_1, X_2\right)$$

is non-increasing in t on $[0,\infty)$.

Given an r.i. space Z on $\mathcal{M}_+([0,\infty))$, for which $\left\|\frac{1}{1+t}\right\|_Z < \infty$, the space X, with $\|x\|_X$ defined at $x \in X_1 + X_2$ by

$$\|x\|_X := \left\|t^{-1} K\left(t, x; X_1, X_2\right)\right\|_Z,$$

satisfies

$$X_1 \cap X_2 \subset X \subset X_1 + X_2.$$

Moreover, for any linear operator T defined on $X_1 + X_2$,

$$T : X_i \to X_i, \quad i = 1, 2, \quad \text{implies} \quad T : X \to X.$$

We say the space X is generated by the K-method of interpolation.

The asserted connection of the duality theory for the K-method with our task is through certain reformulations of (6.1) and (6.2), namely

$$\|f\|_{Y'_X(\Omega)} \approx \left\|t^{\frac{m}{n}-1} K\left(t^{1-\frac{m}{n}}, f; L^{\frac{n}{n-m},\infty}(0,1), L^\infty(0,1)\right)\right\|_{X'(0,1)}, \quad f \in \mathcal{M}(\Omega),$$

and

$$\|g\|_{\widetilde{X}'(\Omega)} \approx \left\|t^{\frac{m}{n}-1} K\left(t^{1-\frac{m}{n}}, g; L^{\frac{n}{n-m},1}(0,1), L^\infty(0,1)\right)\right\|_{X'(0,1)}, \quad g \in \mathcal{M}(\Omega).$$

We finish by an example involving Orlicz spaces.

Theorem 6.3. *Let A be a Young function. Assume that*

$$t^{\frac{m}{n}-1} \notin L^{\widetilde{A}}([0,\infty)).$$

Define B through the equation

$$B(\gamma(t)) := \left(\tfrac{m}{n} - 1\right) \widetilde{A}\left(t^{\frac{m}{n}-1}\right) \frac{\gamma t)}{t\gamma'(t))},$$

in which

$$\gamma(t) := t^{-\frac{m}{n}} \int_t^\infty \widetilde{A}\left(s^{\frac{m}{n}-1}\right) ds, \quad t \in [0,\infty).$$

Define the space $Z(\Omega)$ by

$$\|g\|_{Z(\Omega)} := \left\|t^{\frac{m}{n}-1} \int_0^{t^{1-\frac{m}{n}}} g^*(s)\, ds\right\|_{L^{\widetilde{A}}(0,1)}, \quad g \in \mathcal{M}(\Omega).$$

Then, B is a Young function and

$$\|f\|_{Z'(\Omega)} \approx \left\| t^{-\frac{m}{n}} f^* \left(t^{1-\frac{m}{n}} \right) \right\|_{X(0,1)}, \quad f \in \mathcal{M}(\Omega).$$

It is of interest to compare this result with that of Cianchi who obtained, in [35], a description of $Y_X(\Omega)$ different from ours by the use of techniques specific to the Orlicz context.

7. The gateway to compactness

The most important applications of Sobolev spaces to PDEs require not only embeddings into other function spaces, but, even more frequently, *compact* embeddings.

Let $X(\Omega)$ and $Y(\Omega)$ be two r.i. spaces. We say that $W^m X(\Omega)$ is *compactly embedded* into $Y(\Omega)$, written

$$W^m X(\Omega) \hookrightarrow\hookrightarrow Y(\Omega),$$

if, for every sequence $\{f_k\}$ bounded in $W^m X(\Omega)$, there exists a subsequence $\{f_{k_j}\}$ which is convergent in $Y(\Omega)$.

For the case when $X(\Omega)$ and $Y(\Omega)$ are Lebesgue spaces, we have a theorem, which originated in a lemma of Rellich [101] and was proved specifically for Sobolev spaces by Kondrachov [72], and which asserts that

$$W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \tag{7.1}$$

if $q < p^* = \frac{np}{n-mp}$. Standard examples (see [4]) show it is not compact when $q = \frac{np}{n-mp}$.

As for embeddings into Orlicz spaces, Hempel, Morris and Trudinger [61] showed that the embedding (2.3) is not compact. By a standard argument using a uniform absolute continuity of a norm it can be proved that

$$W^{1,n}(\Omega) \hookrightarrow\hookrightarrow L^B(\Omega)$$

whenever B is a Young function satisfying, with $A(t) := \exp(t^{n'})$ for large values of t,

$$\lim_{t\to\infty} \frac{A(\lambda t)}{B(t)} = \infty$$

for every $\lambda > 0$. Considering Lorentz spaces, it is of interest to notice that even the Sobolev embedding

$$W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^{p^*,\infty}(\Omega)$$

is still not compact. This is not difficult to verify; in fact, standard examples that demonstrate the non-compactness of (7.1) with $q = p^*$ (see e.g. [4]), are sufficient. The space $L^{p^*,\infty}(\Omega)$ is of course considerably larger than $L^{p^*}(\Omega)$, but it simply is not 'larger enough'. This observation is a good point of departure as it raises interesting questions.

For example, we may ask whether the space $L^{p^*,\infty}(\Omega)$ is the *'gateway to compactness'* in the sense that every strictly larger space is already a compact range for $W^{m,p}(\Omega)$. It even makes a good sense to formulate this problem in a broader context of r.i. spaces. Recall that when the Lebesgue space $L^{p^*}(\Omega)$ is replaced by an arbitrary r.i. space $Y(\Omega)$, then the role of $L^{p^*,\infty}(\Omega)$ is taken over by the endpoint Marcinkiewicz space $M_Y(\Omega)$.

We can then formulate the following general question (which we have answered for the particular example above).

Let $X(\Omega)$ be an r.i. space and let $Y_X(\Omega)$ be the corresponding optimal range r.i. space. Let $M_{Y_X}(\Omega)$ be the Marcinkiewicz space corresponding to $Y_X(\Omega)$. Then, of course, we have $W^m X(\Omega) \hookrightarrow M_{Y_X}(\Omega)$. Can this embedding ever be compact? If not, is the Marcinkiewicz space the gateway to compactness in the above-mentioned sense?

It is clear that in order to obtain satisfactory answers to these and other questions we need a reasonable characterization of pairs of spaces $X(\Omega), Y(\Omega)$ for which we have the compact Sobolev embedding

$$W^m X(\Omega) \hookrightarrow \hookrightarrow Y(\Omega).$$

From various analogous results in less general situations it can be guessed that one such characterization might be the compactness of $H_{\frac{n}{m}}$ from $X(0,1)$ to $Y(0,1)$, and another one might be the uniform absolute continuity of the norms of the $H_{\frac{n}{m}}$-image of the unit ball of $X(0,1)$ in $Y(0,1)$. This guess turns out to be reasonable, but the proof is deep and difficult and contains many unexpected pitfalls. Moreover, the case when the range space is $L^\infty(\Omega)$ must be treated separately.

Theorem 7.1. *Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. Assume $Y(\Omega) \neq L^\infty(\Omega)$. Then, the following three statements are equivalent:*

$$W^m X(\Omega) \hookrightarrow \hookrightarrow Y(\Omega); \tag{7.2}$$

$$H_{\frac{n}{m}} : X(0,1) \to\to Y(0,1); \tag{7.3}$$

$$\lim_{a\to 0+} \sup_{\|f\|_{X(0,1)}\le 1} \left\| \chi_{(0,a)}(t) \int_t^1 f^*(s) s^{\frac{m}{n}-1}\, ds \right\|_{Y(0,1)} = 0. \tag{7.4}$$

The case $Y(\Omega) = L^\infty(\Omega)$ is different and, as such, is treated in

Theorem 7.2. *Let $X(\Omega)$ be an r.i. space. Then, the following three statements are equivalent:*

$$W^m X(\Omega) \hookrightarrow \hookrightarrow L^\infty(\Omega);$$
$$H_{\frac{n}{m}} : X(0,1) \to\to L^\infty(0,1);$$
$$\lim_{a\to 0+} \sup_{\|f\|_{X(0,1)}\le 1} \int_0^a f^*(t) t^{\frac{m}{n}-1}\, dt = 0.$$

The most important and involved part is the sufficiency of (7.4) for (7.2). When trying to prove this implication, we discovered an unpleasant technical difficulty. All the methods which we tried to apply, and which would naturally solve the problem, seemed to require

$$Y(0,1) \in \operatorname{Int}\left(L^{\frac{n}{n-m},1}(0,1), L^\infty(0,1)\right),$$

a restriction that does not offer any obvious circumvention. Such requirement, however, is simply too much to ask. A candidate for a compact range can be as large as one wants. For example, since $W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$, $L^1(\Omega)$ is always a compact range for $W^1X(\Omega)$ with any r.i. space $X(\Omega)$. In particular, it may by all means lay far outside from the required interpolation sandwich. This obstacle proved to be surprisingly difficult. At the end, it was overcome by the discovery of a useful fact that, given an r.i. space $Y(\Omega)$, we can always construct another one, $Z(\Omega)$, possibly smaller than $Y(\Omega)$, such that the condition (7.4) is still valid, but which already has the required interpolation properties. We formulate this result as a separate theorem, because it is of independent interest.

Theorem 7.3. *Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces satisfying* (7.4). *Then, there exists another r.i. space, $Z(\Omega)$, with*

$$Z(0,1) \in \operatorname{Int}\left(L^{\frac{n}{n-m},1}(0,1), L^\infty(0,1)\right),$$

such that $Z(\Omega) \hookrightarrow Y(\Omega)$ and

$$\lim_{a\to 0+} \sup_{\|f\|_{X(\Omega)}\le 1} \left\| \chi_{(0,a)}(t) \int_t^1 f^*(s) s^{\frac{m}{n}-1}\, ds \right\|_{Z(\Omega)} = 0.$$

The rest of the proof of the main result uses sharp estimates for supremum operators, various optimality results from the preceding sections and the Arzelà–Ascoli theorem. The proof of Theorem 7.3 is very involved and delicate and requires extensive preparations. The details can be found in [69].

At one stage of the proof, the necessity of the vanishing *Muckenhoupt condition* is shown.

Theorem 7.4. *Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. Assume $Y(\Omega) \ne L^\infty(\Omega)$. Then, each of* (7.2), (7.3) *and* (7.4) *implies*

$$\lim_{a\to 0+} \left\|\chi_{(0,a)}\right\|_{Y(0,1)} \left\| t^{\frac{m}{n}-1} \chi_{(a,1)}(t)\right\|_{X'(0,1)} = 0.$$

This result shows that a candidate $Y(\Omega)$ for a compact range must have an essentially smaller fundamental function than the optimal embedding space $Y_X(\Omega)$, hence also than the Marcinkiewicz space $M_{Y_X}(\Omega)$. In other words, we must have

$$\lim_{t\to 0+} \frac{\varphi_Y(t)}{\varphi_{Y_X}(t)} = 0.$$

This solves one of the above questions: the embedding

$$W^m X(\Omega) \hookrightarrow M_{Y_X}(\Omega)$$

is always true, but *never* (for any choice of $X(\Omega)$) compact.

Likewise, the 'gateway' problem has the negative answer: a counterexample is easily constructed by taking appropriate fundamental functions and using corresponding Marcinkiewicz spaces. It turns out that not even a space which contains $M_{Y_X}(\Omega)$ properly and whose fundamental function is strictly smaller than that of $Y_X(\Omega)$ guarantees compactness.

The connection between a candidate $Y(\Omega)$ for a compact range for a given Sobolev space $W^m X(\Omega)$ and the optimal range $Y_X(\Omega)$ that does imply compactness can be found, but it has to be formulated in terms of a uniform absolute continuity.

Theorem 7.5. *Suppose $X(\Omega)$ and $Y(\Omega)$ are two r.i. spaces. Assume $Y(\Omega) \neq L^\infty(\Omega)$ and let $Y_X(\Omega)$ be the optimal r.i. embedding space for $W^m X(\Omega)$. Then,* (7.2) *holds if and only if the functions in the unit ball of $Y_X(\Omega)$ have uniformly absolutely continuous norms in $Y(\Omega)$ or, what is the same,*

$$\lim_{a \to 0+} \sup_{\|f\|_{Y_X(\Omega)} \leq 1} \left\| \chi_{(0,a)} f^* \right\|_{Y(0,1)} = 0, \quad f \in \mathcal{M}(\Omega). \tag{7.5}$$

Theorem 7.5 gives a necessary and sufficient condition for the compactness of a Sobolev embedding. However, an application of the criterion would involve examination of a uniform absolute continuity of many functions, which may be difficult to verify. It is thus worth looking for a more manageable condition, sufficient for the compactness of the embedding and not too far from being also necessary, which could be used in practical examples. Such a condition is provided by our next theorem. In some sense, it substitutes the negative outcome of the gateway problem.

Theorem 7.6. *Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. Set*

$$\phi_R(t) := \frac{dc}{dt},$$

where $c(t)$ is the least concave majorant of

$$t \|s^{\frac{m}{n}-1} \chi_{(t,1)}(s)\|_{X'(0,1)}.$$

Then, the condition

$$\lim_{a \to 0+} \left\| \chi_{(0,a)} \phi_R \right\|_{Y(0,1)} = 0 \tag{7.6}$$

suffices for

$$W^m X(\Omega) \hookrightarrow \hookrightarrow Y(\Omega).$$

We note that the conclusion of Theorem 7.6 can be reformulated using the special relation between function spaces, called the *almost-compact embedding*, treated in [105], see also [99].

Definition 7.7. Suppose that X and Y are Banach function spaces over a totally σ-finite measure space (R, μ). We say that X is *almost-compactly embedded into* Y and write $X \overset{*}{\hookrightarrow} Y$ if for every sequence $(E_n)_{n=1}^\infty$ of μ-measurable subsets of R satisfying $E_n \to \emptyset$ μ-a.e., we have

$$\lim_{n \to \infty} \sup_{\|f\|_X \leq 1} \|f \chi_{E_n}\|_Y = 0. \tag{7.7}$$

Condition (7.7) says that functions from the unit ball of the space X have uniformly absolutely continuous norms in Y. For this reason almost-compact embeddings are in some literature (see, for example, [53]) also called *absolutely continuous embeddings*.

Corollary 7.8. *Let $X(\Omega)$ and $Y(\Omega)$ be r.i. spaces. Assume that*

$$M_{Y_X} \overset{*}{\hookrightarrow} Y.$$

Then

$$W^m X(\Omega) \hookrightarrow \hookrightarrow Y(\Omega).$$

Observe that the condition (7.6) can be simply verified in concrete examples, since it requires to consider just one function rather than the whole unit ball as in (7.5).

Among many examples that can be extracted from these results, we shall present just one, concerning Orlicz spaces.

Theorem 7.9. *Suppose A and $\tilde{A}$ are complementary Young functions and*

$$\int_1^\infty \frac{\tilde{A}(s)}{s^{1+\frac{n}{n-m}}}\, ds = \infty.$$

Define the Young function $A_R(t)$, for t large, by

$$A_R^{-1}(t) := \frac{t^{1-\frac{m}{n}}}{E^{-1}(t)},$$

with

$$E(t) := t^{\frac{n}{n-m}} \int_1^t \frac{\tilde{A}(s)}{s^{1+\frac{n}{n-m}}}\, ds, \quad t \geq 1.$$

Then,

$$W^m L^A(\Omega) \hookrightarrow \hookrightarrow L^B(\Omega)$$

for a given Young function B if and only if

$$\lim_{t\to\infty} \frac{A_R(\lambda t)}{B(t)} = \infty, \tag{7.8}$$

for every $\lambda > 0$.

We finally note that in terms of the explicitly known functions B and E, (7.8) can be expressed by

$$\lim_{t\to\infty} \frac{B\left((\lambda t)^{-1} E(t)^{1-\frac{m}{n}}\right)}{E(t)} = 0, \quad \text{for every } \lambda > 0.$$

8. Boundary traces

One of the main applications of Sobolev space techniques is in the field of traces of functions defined on domains. The theory of boundary traces in Sobolev spaces has a number of applications, especially to boundary-value problems for partial differential equations, in particular when the Neumann problem is studied. The *trace operator*, defined by

$$\mathrm{Tr}\ u = u_{|\partial\Omega}$$

for a continuous function u on $\overline{\Omega}$, where $\partial\Omega$ is the *boundary* of Ω, can be extended to a bounded linear operator

$$\mathrm{Tr} : W^{1,1}(\Omega) \to L^1(\partial\Omega),$$

where $L^1(\partial\Omega)$ denotes the Lebesgue space of summable functions on $\partial\Omega$ with respect to the $(n-1)$–dimensional Hausdorff measure $\mathcal{H}^{n-1}$.

The theory of boundary traces in Sobolev spaces has been developed, via different methods and in different settings, by various authors, including Besov [16], Gagliardo [54], Lions and Magenes [79]. Extensions of this theory to the case where the restriction of $\mathcal{H}^{n-1}$ to $\partial\Omega$ is replaced by more general measures in $\overline{\Omega}$ have also been extensively studied, see for example the survey papers [85, 114] or the monographs [5, 83, 91, 117].

Assume that $1 \leq m < n$. The standard trace embedding theorem (cf. e.g. [1]) tells us that if $1 \leq p < \frac{n}{m}$, then a constant $C = C(\Omega, p, m)$ exists such that

$$\|\operatorname{Tr} u\|_{L^{\frac{p(n-1)}{n-mp}}(\partial\Omega)} \leq C\|u\|_{W^{m,p}(\Omega)} \tag{8.1}$$

for every $u \in W^{m,p}(\Omega)$, whereas, if $p > \frac{n}{m}$, then a constant $C = C(\Omega, p, m)$ exists such that

$$\|\operatorname{Tr} u\|_{L^\infty(\partial\Omega)} \leq C\|u\|_{W^{m,p}(\Omega)} \tag{8.2}$$

for every $u \in W^{m,p}(\Omega)$.

A classical approach to trace inequalities such as (8.1) and (8.2) makes use of local coordinates on $\partial\Omega$ and of Sobolev inequalities in Ω, and heavily relies upon the fact that Lebesgue norms of $\operatorname{Tr} u$ and of $D^m u$ are involved. On the other hand, such a method does not seem to allow extensions to trace inequalities when more general norms are in play. Alternative techniques can be used to deal with specific instances. For example, a representation formula for Sobolev functions in terms of Riesz potentials enables one to effectively treat the limiting case when $p = \frac{n}{m}$, and to prove that a constant $C = C(\Omega, m)$ exists such that

$$\|\operatorname{Tr} u\|_{\exp L^{\frac{n}{n-m}}(\partial\Omega)} \leq C\|u\|_{W^{m,\frac{n}{m}}(\Omega)} \tag{8.3}$$

for every $u \in W^{m,\frac{n}{m}}(\Omega)$ (see e.g. [5, 7.6.4]). The shortcoming of such an approach is that function spaces defined in terms of potentials are not always equivalent to corresponding spaces defined by derivatives. In fact, none of the available methods seems to cover the whole range of situations of interest in applications.

Thus, the problem can be raised of a unified flexible approach to trace inequalities fitting a fairly general class of function spaces. A powerful tool in the study of (first-order) Sobolev type inequalities in the whole domain Ω for functions vanishing on $\partial\Omega$ is symmetrization. The strength of this elegant technique, which relies upon the Pólya–Szegö principle on the decrease of gradient norms under radially non-increasing rearrangement ([23, 65, 111]) and has led to such results as those of Aubin [7] and Talenti [111], and of Moser [86], is in that the relevant Sobolev inequalities are reduced to one-dimensional inequalities.

In [38], a method was developed for obtaining sharp trace inequalities in a general context based on the ideas elaborated in the preceding sections. Again, the key result is a reduction theorem.

Theorem 8.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, having a Lipschitz boundary. Let $X(\Omega)$ and $Y(\partial\Omega)$ be r.i. spaces. Then*

$$\|\operatorname{Tr} u\|_{Y(\partial\Omega)} \leq C\|u\|_{W^m X(\Omega)} \tag{8.4}$$

for every $u \in W^m X(\Omega)$ if and only if

$$\left\| \int_{t^{n'}}^1 f(s) s^{\frac{m}{n}-1}\, ds \right\|_{Y(0,1)} \leq C\|f\|_{X(0,1)}, \quad f \in \mathcal{M}_+(0,1),$$

.

Thus, when dealing with boundary traces, the role of the operator $H_{\frac{n}{m}}$ is taken over by the operator $\int_{t^{n'}}^{1} f(s)s^{\frac{m}{n}-1}\,ds$. As a next step, we can characterize the optimal trace range on $\partial\Omega$.

Theorem 8.2. *Let Ω, n and m be as in* Theorem 8.1. *Let $X(\Omega)$ be an r.i. space. Then, the r.i. space $Y(\partial\Omega)$, whose associate norm is given by*

$$\|g\|_{Y'(\partial\Omega)} = \left\| t^{\frac{m-1}{n}} g^{**}(t^{\frac{1}{n'}}) \right\|_{X'(0,1)}$$

for every $\mathcal{H}^{n-1}$–measurable function g on $\partial\Omega$, is optimal in (8.4).

Our trace results recover many known examples, prove their optimality that had not been known before, and bring new ones.

Theorem 8.3. *Let Ω, n and m be as in* Theorem 8.1.

(i) *If $1 \le p < \frac{n}{m}$ and $1 \le q \le \infty$, then a constant $C = C(\Omega, p, q, m)$ exists such that*

$$\|\operatorname{Tr} u\|_{L^{\frac{p(n-1)}{n-mp},q}(\partial\Omega)} \le C\|u\|_{W^m L^{p,q}(\Omega)} \tag{8.5}$$

for every $u \in W^m L^{p,q}(\Omega)$.

(ii) *If $p = \frac{n}{m}$ and $1 < q \le \infty$, then a constant $C = C(\Omega, q, m)$ exists such that*

$$\|\operatorname{Tr} u\|_{L^{\infty,q;-1}(\partial\Omega)} \le C\|u\|_{W^m L^{\frac{n}{m},q}(\Omega)} \tag{8.6}$$

for every $u \in W^m L^{\frac{n}{m}}(\Omega)$.

(iii) *If either $p = \frac{n}{m}$ and $q = 1$ or $p > \frac{n}{m}$ and $1 \le q \le \infty$, then a constant $C = C(\Omega, p, q, m)$ exists such that*

$$\|\operatorname{Tr} u\|_{L^{\infty}(\partial\Omega)} \le C\|u\|_{W^m L^{p,q}(\Omega)} \tag{8.7}$$

for every $u \in W^{m,p}(\Omega)$.

The spaces $L^{\frac{p(n-1)}{n-mp},q}(\partial\Omega)$, $L^{\infty,q}(\log L)^{-1}(\partial\Omega)$ and $L^{\infty}(\partial\Omega)$ are optimal among all r.i. spaces on $\partial\Omega$ in (8.5), (8.6) *and* (8.7), *respectively.*

It is worth to study the traces of Sobolev functions defined on more general subsets of Ω than just the boundary. Then, the dimension of the subset comes to the picture and the problem becomes rather subtle. One important property enjoyed by functions from the Sobolev space $W^{m,p}(\mathbb{R}^n)$, $m \in \mathbb{N}$, $p \in [1, \infty]$, is that their traces to lower dimensional spaces can be *properly defined*, provided that the dimension d of the relevant subspaces is *not too small*, depending on n, m and p. The trace of a function $u \in W^{m,p}(\mathbb{R}^n)$ turns out to be measurable with respect to the d-dimensional measure on the relevant subspaces, and also integrable to some power q, depending on n, m, p and d. Loosely speaking, increasing the values of m and p causes u to be more regular, and hence allows smaller values of d and larger values of q. To be more specific, let $n, d \in \mathbb{N}$, and let $n \ge 2$ and $1 \le d < n$. Since $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$, any point $x \in \mathbb{R}^n$ can be represented as $x = (y, z)$, with $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^{n-d}$. Moreover, $\mathbb{R}^d$ can be identified with the subspace of those points in $\mathbb{R}^n$ having the form $(y, 0)$ for some $y \in \mathbb{R}^d$.

Various approaches to traces of functions are available in the literature. We shall adopt the following definition, which extends more customary notions of traces of functions in Sobolev spaces – see [27, Chapter 5]. A function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to have a trace $\operatorname{Tr} u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ on $\mathbb{R}^d$ if there exists a function $\overline{u}$, equivalent to u on $\mathbb{R}^n$, such that

$$\lim_{z\to 0} \overline{u}(\cdot, z) = \operatorname{Tr} u(\cdot) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d).$$

A standard trace embedding theorem (see e.g. [1, Theorem 5.4] or [83, Corollary 1.4.1]), combined with [27, Corollary 1, Chapter 5], tells us that if $1 \le m < n$ and either

$$d \ge n - m \quad \text{and} \quad p \ge 1, \tag{8.8}$$

or

$$d < n - m \quad \text{and} \quad p > \frac{n-d}{m}, \tag{8.9}$$

then every function $u \in W^{m,p}_{\mathrm{loc}}(\mathbb{R}^n)$ has a trace on $\mathbb{R}^d$. Moreover, the operator Tr, which associates $\operatorname{Tr} u$ with u, is linear, and, if $p < \frac{n}{m}$, then

$$\operatorname{Tr} : W^{m,p}(\mathbb{R}^n) \to L^{\frac{dp}{n-mp}}(\mathbb{R}^d), \tag{8.10}$$

where the arrow "$\to$" stands for boundedness of the operator. In particular,

$$\|\operatorname{Tr} u\|_{L^{\frac{dp}{n-mp}}(\mathbb{R}^d)} \le C\|\nabla^m u\|_{L^p(\mathbb{R}^n)} \tag{8.11}$$

for every $u \in W^{m,p}(\mathbb{R}^n)$. Note that the case when $m \ge n$ is uninteresting, since any function $u \in W^{m,p}_{\mathrm{loc}}(\mathbb{R}^n)$, $p \ge 1$, is continuous, and hence $\operatorname{Tr} u$ trivially exists on $\mathbb{R}^d$ for every $d \in [1, n-1]$.

Unlike (8.8), in the limiting case when $p = \frac{n-d}{m} > 1$, functions from the Sobolev space $W^{m,\frac{n-d}{m}}_{\mathrm{loc}}(\mathbb{R}^n)$ need not admit a trace on $\mathbb{R}^d$. Our next goal is to fill in this gap and to exhibit an optimal (largest possible) Sobolev-type space when $d < n-m$ such that all of its functions admit a trace on $\mathbb{R}^d$. It was shown in [42] that the existence of traces can be restored when $p = \frac{n-d}{m}$, provided that $W^{m,\frac{n-d}{m}}_{\mathrm{loc}}(\mathbb{R}^n)$ is replaced by the Sobolev type space $W^m_{\mathrm{loc}}L^{\frac{n-d}{m},1}(\mathbb{R}^n)$ built upon the Lorentz space $L^{\frac{n-d}{m},1}(\mathbb{R}^n)$. Such a space is slightly smaller than $W^{m,\frac{n-d}{m}}_{\mathrm{loc}}(\mathbb{R}^n)$ if $\frac{n-d}{m} > 1$, but agrees with the standard space $W^{m,1}_{\mathrm{loc}}(\mathbb{R}^n)$ when $\frac{n-d}{m} = 1$. Moreover, $L^{\frac{n-d}{m},1}_{\mathrm{loc}}(\mathbb{R}^n)$ is optimal among all rearrangement-invariant spaces.

Observe that, since $L^{p,q}(\mathbb{R}^n) \subsetneq L^{p,r}(\mathbb{R}^n)$ if $1 \le q < r \le \infty$, and $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, actually

$$W^m L^{\frac{n-d}{m},1}(\mathbb{R}^n) \subsetneq W^{m,\frac{n-d}{m}}(\mathbb{R}^n) \qquad \text{if } \tfrac{n-d}{m} > 1, \tag{8.12}$$

whereas

$$W^m L^{1,1}(\mathbb{R}^n) = W^{m,1}(\mathbb{R}^n). \tag{8.13}$$

Relations (8.12) and (8.13) continue to hold if all the Sobolev spaces are replaced by their local versions.

Theorem 8.4. *Assume that $n \ge 2$, $1 \le m < n$ and $1 \le d \le n - m$. Then any function from $W^m_{\mathrm{loc}}L^{\frac{n-d}{m},1}(\mathbb{R}^n)$ admits a trace on $\mathbb{R}^d$. Moreover, $L^{\frac{n-d}{m},1}_{\mathrm{loc}}(\mathbb{R}^n)$ is the optimal (largest) rearrangement-invariant space enjoying this property, in the sense* *that if $X(\mathbb{R}^n)$ is another r.i. space such that any function from $W^m_{\text{loc}}X(\mathbb{R}^n)$ admits a trace on $\mathbb{R}^d$, then, necessarily, $X_{\text{loc}}(\mathbb{R}^n) \subset L^{\frac{n-d}{m},1}_{\text{loc}}(\mathbb{R}^n)$.*

In an analogy with the classical situation described in (8.10)–(8.11), we establish a trace embedding for $W^mL^{\frac{n-d}{m},1}(\mathbb{R}^n)$. In fact, we find the optimal (smallest) range space in the class of Lorentz spaces for trace embeddings of $W^mL^{\frac{n-d}{m},1}(\mathbb{R}^n)$. Interestingly enough, the optimal range space in this endpoint trace embedding turns out not to be the genuine Lorentz space $L^{\frac{n-d}{m},1}(\mathbb{R}^d)$ as one would expect in the light of other known optimal Sobolev embeddings such as those treated in [88, 93, 22, 59] (see [48, 50] for the optimality), and trace embeddings ([38]), but merely the (strictly larger) Lebesgue space $L^{\frac{n-d}{m}}(\mathbb{R}^d)$.

Theorem 8.5. *Assume that $n \geq 2$, $1 \leq m < n$ and $1 \leq d \leq n-m$. Then*

$$\operatorname{Tr} : W^mL^{\frac{n-d}{m},1}(\mathbb{R}^n) \to L^{\frac{n-d}{m}}(\mathbb{R}^d). \tag{8.14}$$

In particular, a constant $C = C(n,m,d)$ exists such that

$$\|\operatorname{Tr} u\|_{L^{\frac{n-d}{m}}(\mathbb{R}^d)} \leq C\|\nabla^m u\|_{L^{\frac{n-d}{m},1}(\mathbb{R}^n)} \tag{8.15}$$

for every $u \in W^mL^{\frac{n-d}{m},1}(\mathbb{R}^n)$. Moreover, $L^{\frac{n-d}{m}}(\mathbb{R}^d)$ is optimal on the left-hand side of (8.15) *among all Lorentz spaces.*

Let us mention that a result on a related topic has recently been established in [46], where a characterization of Sobolev inequalities involving general measures and Lorentz norms is given in terms of capacitary inequalities.

Embedding (8.14) continues to hold provided that the whole of $\mathbb{R}^n$ is replaced by any extension domain Ω (see e.g. [1, 27, 108, 117] for a definition). Of course, $\mathbb{R}^d$ has to be replaced by $\Omega \cap \mathbb{R}^d$ in this case.

Another generalization of embedding (8.14) concerns the case when traces on d-dimensional subspaces are replaced by (suitably defined – see e.g. [27]) traces on smooth d-dimensional Riemannian submanifolds of $\mathbb{R}^n$.

Our next aim is to formulate higher-order optimal trace theorems for specific function spaces. The rest of this section contains results from [43].

We define the subspace $W^m_\perp X(\Omega)$ of $W^mX(\Omega)$ as

$$W^m_\perp X(\Omega) = \left\{u \in W^mX(\Omega) : \int_\Omega \nabla^k u\,dx = 0, \text{ for } 0 \leq k \leq m-1\right\}. \tag{8.16}$$

Theorem 8.6. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $m \in \mathbb{N}$ and $d \in \mathbb{N}$ be such that $1 \leq d \leq n$ and $n - d \leq m$. Let Ω_d be the (non empty) intersection of Ω with any d-dimensional affine subspace of $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The Sobolev trace embedding*

$$\operatorname{Tr} : W^mX(\Omega) \to Y(\Omega_d) \tag{8.17}$$

holds. Here, $X(\Omega)$ denotes the rearrangement-invariant space on Ω associated with $\|\cdot\|_{X(0,1)}$, and $Y(\Omega_d)$ the rearrangement-invariant space on Ω_d, with respect to the d-dimensional Hausdorff measure $\mathcal{H}^d$, associated with $\|\cdot\|_{Y(0,1)}$.

(ii) *The Poincaré trace inequality*

$$\|\mathrm{Tr}\, u\|_{Y(\Omega_d)} \leq C_1 \, \|\nabla^m u\|_{X(\Omega)} \tag{8.18}$$

holds for some constant C_1, and for every $u \in W^m_\perp X(\Omega)$.

(iii) *The inequality*

$$\left\| \int_{t^{\frac{n}{d}}}^{1} f(s)\, s^{-1+\frac{m}{n}}\, ds \right\|_{Y(0,1)} \leq C_2 \, \|f\|_{X(0,1)} \tag{8.19}$$

holds for some constant C and for every $f \in X(0,1)$.

A characterization of the optimal target space $Y(\Omega_d)$ in the trace embedding (8.17), for any given rearrangement-invariant space $X(\Omega)$ is provided by the next result. Its statement requires the following definition. Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, and $n, m, d \in \mathbb{N}$ be such that $1 \leq d \leq n$ and $n - d \leq m$, we call $\|\cdot\|_{X^m_{d,n}(0,1)}$ the rearrangement-invariant function norm whose associate function norm is given by

$$\|f\|_{(X^m_{d,n})'(0,1)} = \left\| s^{-1+\frac{m}{n}} \int_0^{s^{\frac{d}{n}}} f^*(r) dr \right\|_{X'(0,1)} \tag{8.20}$$

for $f \in \mathcal{M}(0,1)$.

Theorem 8.7. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $m, d \in \mathbb{N}$ be such that $1 \leq d \leq n$ and $n - d \leq m$. Let Ω_d be the (non empty) intersection of Ω with any d-dimensional affine subspace of $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then*

$$\mathrm{Tr} : W^m X(\Omega) \to X^m_{d,n}(\Omega_d). \tag{8.21}$$

Here, $X(\Omega)$ denotes the rearrangement-invariant space on Ω associated with $\|\cdot\|_{X(0,1)}$, and $X^m_{d,n}(\Omega_d)$ the rearrangement-invariant space on Ω_d associated with $\|\cdot\|_{(X^m_{d,n})'(0,1)}$. Moreover, the space $X^m_{d,n}(\Omega_d)$ is optimal in (8.21) *among all rearrangement-invariant spaces.*

Corollary 8.8. *Let n, d, m, Ω, Ω_d and $\|\cdot\|_{X(0,1)}$ be as in Theorem 8.7. Then the following facts are equivalent:*

$$\mathrm{Tr} : W^m X(\Omega) \to L^\infty(\Omega_d); \tag{8.22}$$

$$X^m_{d,n}(\Omega_d) = L^\infty(\Omega_d); \tag{8.23}$$

$$\|s^{-1+\frac{m}{n}}\|_{X'(0,1)} < \infty. \tag{8.24}$$

In particular, (8.22) *and* (8.23) *hold for any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, provided that $m \geq n$.*

We shall now state the sharpness of the iteration of optimal targets. Let us emphasize that, in this framework, iteration not only involves the order of differentiation, but also the dimension of subspaces.

Theorem 8.9. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $k, h, d, \ell \in \mathbb{N}$, be such that $1 \leq d \leq \ell \leq n$, $n - \ell \leq k$ and $\ell - d \leq h$. Let Ω_ℓ be the (non empty) intersection of Ω with any ℓ-dimensional affine subspace S_ℓ of $\mathbb{R}^n$, and let Ω_d be the (non empty) intersection of Ω with any d-dimensional affine subspace of S_ℓ.*

Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then

$$(X^k_{\ell,n})^h_{d,\ell}(\Omega_d) = X^{k+h}_{d,n}(\Omega_d). \tag{8.25}$$

Theorem 8.10. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $m \in \mathbb{N}$ and $d \in \mathbb{N}$, be such that $1 \leq d \leq n$ and $n - d \leq m$. Let Ω_d be the (non empty) intersection of Ω with any d-dimensional affine subspace of $\mathbb{R}^n$. Assume that either $p = q = 1$ or $p = q = \infty$ or $p \in (1, \infty)$ and $q \in [1, \infty]$. Then*

$$\mathrm{Tr} : W^m L^{p,q}(\Omega) \to \begin{cases} L^{\frac{pd}{n-mp},q}(\Omega_d) & \text{if } m < n \text{ and } p \in [1, \frac{n}{m}), \\ L^{\infty,q;-1}(\Omega_d) & \text{if } m < n,\ p = \frac{n}{m} \text{ and } q > 1, \\ L^\infty(\Omega_d) & \text{otherwise.} \end{cases} \tag{8.26}$$

Moreover, the target spaces in (8.26) *are optimal among all rearrangement-invariant spaces on Ω_d.*

Let us now focus on Sobolev trace embeddings in Orlicz spaces.

Let $n, m, d \in \mathbb{N}$ be as in the statement of Theorem 8.6. Let A be a Young function such that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{m}{n-m}} dt < \infty. \tag{8.27}$$

If $m < n$, assume that the integral

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{m}{n-m}} dt \tag{8.28}$$

diverges, let $H_m : [0, \infty) \to [0, \infty)$ be the function defined as

$$H_m(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{m}{n-m}} dt\right)^{\frac{n}{n-m}} \qquad \text{for } s \geq 0, \tag{8.29}$$

and let $A_{m,d}$ be the Young function given by

$$A_{m,d}(t) = \int_0^{H_m^{-1}(t)} \left(\frac{A(s)}{s}\right)^{\frac{d-m}{n-m}} H_m(s)^{\frac{d-n}{n-m}} ds \qquad \text{for } t \geq 0. \tag{8.30}$$

Theorem 8.11. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $m \in \mathbb{N}$ and $d \in \mathbb{N}$ be such that $1 \leq d \leq n$ and $n - d \leq m$. Let Ω_d be the (non empty) intersection of Ω with any d-dimensional affine subspace of $\mathbb{R}^n$. Let A be a Young function fulfilling* (8.27). *Then*

(8.31)
$$\mathrm{Tr} : W^m L^A(\Omega) \to \begin{cases} L^{A_{m,d}}(\Omega_d) & \textit{if } m < n, \textit{ and the integral } (8.28) \textit{ diverges,} \\ L^\infty(\Omega_d) & \textit{if either } m \geq n, \textit{ or } m < n \\ & \textit{and the integral } (8.28) \textit{ converges.} \end{cases}$$

Moreover, the target spaces in (8.31) *are optimal among all Orlicz spaces.*

Theorem 8.11 follows from Theorem 8.6 and [36, Theorem 3.5].

It turns out that the optimal rearrangement-invariant target space in the first case of (8.31) is an *Orlicz-Lorentz space*. Under the assumption that $m < n$ and that the integral in (8.28) is divergent, let $a : [0, \infty) \to [0, \infty]$ be a non-decreasing, left-continuous function which is neither identically equal to 0 nor to ∞ and which satisfies

$$A(t) = \int_0^t a(\tau) d\tau \qquad \text{for } t \geq 0, \tag{8.32}$$

let E_m be the Young function given by

$$E_m(t) = \int_0^t e_m(s) ds \qquad \text{for } t \geq 0, \tag{8.33}$$

where e_m is the non-decreasing, left-continuous function in $[0, \infty)$ obeying

$$e_m^{-1}(s) = \left(\int_{a^{-1}(s)}^{\infty} \left(\int_0^{\tau} \left(\frac{1}{a(t)} \right)^{\frac{m}{n-m}} dt \right)^{-\frac{n}{m}} \frac{d\tau}{a(\tau)^{\frac{n}{n-m}}} \right)^{\frac{m}{m-n}} \qquad \text{for } s \geq 0\,. \tag{8.34}$$

We define the *Orlicz-Lorentz space* $L(\frac{n}{m}, \frac{n}{d}, E_m)(\Omega_d)$ as the collection of measurable functions having finite norm

$$\|g\|_{L(\frac{n}{m}, \frac{n}{d}, E_m)(\Omega_d)} = \|s^{-\frac{m}{n}} g^*(\mathcal{H}^d(\Omega_d) s^{\frac{d}{n}})\|_{L^{E_m}(0,1)} \tag{8.35}$$

for any appropriate function g.

Theorem 8.12. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $m \in \mathbb{N}$ and $d \in \mathbb{N}$ be such that $1 \leq d \leq n$ and $n - d \leq m$. Let Ω_d be the (non empty) intersection of Ω with any d-dimensional affine subspace of $\mathbb{R}^n$. Let A be a Young function fulfilling* (8.27)*. Assume that $m < n$, and the integral in* (8.28) *diverges. Then*

$$\mathrm{Tr} : W^m L^A(\Omega) \to L(\tfrac{n}{m},\, \tfrac{n}{d}, E_m)(\Omega_d), \tag{8.36}$$

and the target space in (8.36) *is optimal among all rearrangement-invariant spaces.*

Embedding (8.36) follows from Theorem 8.6, via an analogous argument as in the proof of [36, Theorem 4.1].

The proof of the main reduction theorem is based on deep properties of several operators involving suprema, on an iteration technique, and on the following result, perhaps of independent interest.

Theorem 8.13. *Assume that $\alpha, \beta, \gamma, \delta \in (0, \infty)$ are such that*

$$\gamma + \delta \geq 1, \quad \alpha + \beta \geq 1, \quad \gamma < 1 \quad \text{and} \quad \alpha + \beta\gamma < 1. \tag{8.37}$$

Then there exists a positive constant A, depending only on α, β, γ and δ such that, for every r.i. function norm $\|\cdot\|_{X(0,1)}$ and every $g \in \mathcal{M}(0,1)$, one has

$$\left\| t^{\alpha-1} \int_0^{t^\beta} \left[\tau^{\gamma-1} \int_0^{\tau^\delta} g^*(r)\, dr \right]^* (s)\, ds \right\|_{X(0,1)} \leq A \left\| t^{\alpha+\beta\gamma-1} \int_0^{t^{\beta\delta}} g^*(s)\, ds \right\|_{X(0,1)}. \tag{8.38}$$

9. Gaussian Sobolev embeddings

Some specific problems in physics such as quantum fields and hypercontractivity semigroups lead to the study of classical Sobolev embeddings in infinitely many variables. The study of quantum fields can be under certain circumstances reduced to operator or semigroup estimates which are in turn equivalent to such inequalities (see [89] and the references therein). However, when we let $n \to \infty$, then $\frac{np}{n-p} \to p+$, and so the gain in integrability would apparently be lost. An even worse problem is the fact that the Lebesgue measure on an infinite-dimensional space is meaningless.

These obstacles were overcome in the fundamental paper of L. Gross [57], who replaced the Lebesgue measure by the Gauss one. Note that the Gauss measure γ is defined on $\mathbb{R}^n$ by

$$d\gamma(x) = (2\pi)^{-\frac{n}{2}}\, e^{\frac{-|x|^2}{2}}\, dx.$$

Now, $\gamma(\mathbb{R}^n) = 1$ for every $n \in \mathbb{N}$, hence the extension for $n \to \infty$ is meaningful. The idea is then to seek a version of the Sobolev inequality that would hold on the probability space $(\mathbb{R}^n, \gamma)$ with a constant independent of n. Gross proved in [57] an inequality of this kind, which, in particular, entails that

$$\|u - u_\gamma\|_{L^2 \log L(\mathbb{R}^n,\gamma)} \le C\|\nabla u\|_{L^2(\mathbb{R}^n,\gamma)} \tag{9.1}$$

for every weakly differentiable function u making the right-hand side finite, where

$$u_\gamma = \int_{\mathbb{R}^n} u(x) d\gamma(x),$$

the mean value of u, and $L^2 \log L(\mathbb{R}^n, \gamma)$ is the Orlicz space (or exponential Zygmund class) of those functions u such that $|u|^2 |\log |u||$ is integrable in $\mathbb{R}^n$ with respect to γ. Interestingly, (9.1) still provides some slight gain in integrability from $|\nabla u|$ to u, even though it is no longer a power-gain.

Gross' result ignited an extensive research on Sobolev inequalities in the Gauss space, including simplified proofs [3], applications [52, 102, 115], extensions to the case when $|\nabla u|$ belongs to a space different from $L^2(\mathbb{R}^n, \gamma_n)$ [6, 8, 12, 9, 20, 19, 45, 76, 94]. For instance, inequalities for functions with $|\nabla u| \in L^p(\mathbb{R}^n, \gamma_n)$ for $p \in [1, \infty)$ are known [2], and tell us that then $u \in L^p \log L^{\frac{p}{2}}(\mathbb{R}^n, \gamma_n)$. Interestingly, in contrast to the Euclidean setting, when $|\nabla u|$ enjoys a high degree of integrability, stronger than just a power, there is a loss of integrability from $|\nabla u|$ to u instead of a gain in the Gaussian Sobolev embedding. This happens, in particular, when $|\nabla u|$ is exponentially integrable [17], or essentially bounded [6]: for instance, in the latter case, one can just infer that $u \in \exp L^2(\mathbb{R}^n, \gamma_n)$, the Orlicz space associated with the Young function $e^{t^2} - 1$. This phenomenon can be explained by the rapid decay of the Gauss measure at infinity.

In [41], we studied problems concerning optimality of function spaces in first-order Sobolev embeddings on the Gaussian space, namely

$$\|u - u_\gamma\|_{Y(\mathbb{R}^n,\gamma)} \le C\|\nabla u\|_{X(\mathbb{R}^n,\gamma)} \tag{9.2}$$

As usual, we start with a reduction theorem. This time, the role of the operator $H_{\frac{n}{m}}$ is taken by the operator

$$\int_t^1 \frac{f(s)}{s\sqrt{1+\log(1/s)}}\,ds.$$

We define the Sobolev space $V^1X(\mathbb{R}^n,\gamma_n)$ built upon $X(\mathbb{R}^n,\gamma_n)$ by

$$\begin{aligned} V^1X(\mathbb{R}^n,\gamma_n) \quad = \quad \{u;\ & u \text{ is a weakly differentiable function on } \mathbb{R}^n \\ & \text{such that } |\nabla u| \in X(\mathbb{R}^n,\gamma_n)\}. \end{aligned}$$

The reduction theorem then reads as follows.

Theorem 9.1. *Let $X(\mathbb{R}^n,\gamma)$ and $Y(\mathbb{R}^n,\gamma)$ be r.i. spaces. Then,*

$$\|u-u_\gamma\|_{Y(\mathbb{R}^n,\gamma)} \le C\,\|\nabla u\|_{X(\mathbb{R}^n,\gamma)}$$

for every $u \in V^1X(\mathbb{R}^n,\gamma)$ if and only if

$$\left\|\int_t^1 \frac{f(s)}{s\sqrt{1+\log(1/s)}}\,ds\right\|_{Y(0,1)} \le C\,\|f\|_{X(0,1)}$$

for every $f \in X(0,1)$.

Then, the characterization of the optimal range r.i. space for the Gaussian Sobolev embedding when the domain space is obtained via the usual scheme.

Theorem 9.2. *Let $X(\mathbb{R}^n,\gamma)$ be an r.i. space, and let $Z(\mathbb{R}^n,\gamma)$ be the r.i. space equipped with the norm*

$$\|g\|_{Z(\mathbb{R}^n,\gamma)} := \left\|\frac{g^{**}(s)}{\sqrt{1+\log\frac{1}{s}}}\right\|_{X'(0,1)}$$

for any measurable function u on $\mathbb{R}^n$. Let $Y(\mathbb{R}^n,\gamma) = Z'(\mathbb{R}^n,\gamma)$. Then $Y(\mathbb{R}^n,\gamma)$ is the optimal range space in the Gaussian Sobolev embedding (9.2).

The role of the operator $T_{\frac{n}{m}}$ is in the Gaussian setting taken over by the operator

$$(Tf)(t) = \sqrt{1+\log\frac{1}{t}}\ \sup_{t\le s\le 1} \frac{f^*(s)}{\sqrt{1+\log\frac{1}{s}}}, \qquad \text{for } t \in (0,1).$$

With the help of the operator T, we can characterize the optimal domain space.

Theorem 9.3. *Let $Y(\mathbb{R}^n,\gamma)$ be an r.i. space such that:*

$$\exp L^2(\mathbb{R}^n,\gamma) \hookrightarrow Y(\mathbb{R}^n,\gamma) \hookrightarrow L\,(\log L)^{\frac{1}{2}}\,(\mathbb{R}^n,\gamma),$$

and

$$T \text{ is bounded on } Y'(0,1).$$

Let $X(\mathbb{R}^n,\gamma)$ be the r.i. space equipped with the norm

$$\|u\|_{X(\mathbb{R}^n,\gamma)} = \left\|\int_t^1 \frac{u^*(s)}{s\sqrt{1+\log\frac{1}{s}}}\,ds\right\|_{Y(0,1)}.$$

Then $X(\mathbb{R}^n,\gamma)$ is the optimal domain space for $Y(\mathbb{R}^n,\gamma)$ in the Gaussian Sobolev embedding (9.2)*.*

We shall now collect the basic examples.

Example 9.4. (i) Let $1 \le p < \infty$. Then the spaces $X(\mathbb{R}^n,\gamma) = L^p(\mathbb{R}^n,\gamma)$, $Y(\mathbb{R}^n,\gamma) = L^p(\log L)^{\frac{p}{2}}(\mathbb{R}^n,\gamma)$ form an optimal pair in the Gaussian Sobolev embedding (9.2).

(ii) The spaces $X(\mathbb{R}^n,\gamma) = L^\infty(\mathbb{R}^n,\gamma), Y(\mathbb{R}^n,\gamma) = \exp L^2(\mathbb{R}^n,\gamma)$ form an optimal pair in the Gaussian Sobolev embedding (9.2).

(iii) Let $\beta > 0$. Then the spaces $(\exp L^\beta(\mathbb{R}^n,\gamma), \exp L^{\frac{2\beta}{2+\beta}}(\mathbb{R}^n,\gamma))$ form an optimal pair in the Gaussian Sobolev embedding (9.2).

We shall now turn our attention to examples involving Orlicz spaces. Let A be a Young function. The function $\widetilde{A} : [0,\infty) \to [0,\infty)$, defined by

$$\widetilde{A}(t) = \sup\{st - A(s) : s \ge 0\} \qquad \text{for } t \in [0,\infty),$$

is also a Young function, called the *Young conjugate* of A. We may assume, without loss of generality, that

$$\int_0 \frac{\widetilde{A}(t)}{t^2}\,dt < \infty\,. \tag{9.3}$$

Actually, A can be replaced, if necessary, by a Young function equivalent near infinity and fulfilling (9.3), without changing $L^A(\mathbb{R}^n,\gamma_n)$ (up to equivalent norms).

Let $E : (0,\infty) \to [0,\infty)$ be the (non-decreasing) function obeying

$$E^{-1}(t) = \left\| \frac{1}{r\sqrt{1+\log_+ \frac{1}{r}}} \right\|_{L^{\widetilde{A}}(\frac{1}{t},\infty)} \qquad \text{for } t > 0, \tag{9.4}$$

where $\log_+ t = \max\{\log t, 0\}$. Note that the right-hand side of (9.4) is actually finite for $t > 0$, owing to (9.3). Define $A_G : [0,\infty) \to [0,\infty)$ by

$$A_G(t) = \int_0^t \frac{E(s)}{s}\,ds \qquad \text{for } t > 0. \tag{9.5}$$

The main result of this section tells us that $L^{A_G}(\mathbb{R}^n,\gamma_n)$ is the optimal Orlicz space into which $V^1L^A(\mathbb{R}^n,\gamma_n)$ is continuously embedded.

Theorem 9.5. *Let A be a Young function (modified, if necessary, near 0 in such a way that* (9.3) *is satisfied). Then, an absolute constant C exists such that*

$$\|u - u_{\gamma_n}\|_{L^{A_G}(\mathbb{R}^n,\gamma_n)} \le C\|\nabla u\|_{L^A(\mathbb{R}^n,\gamma_n)} \tag{9.6}$$

for every $u \in V^1L^A(\mathbb{R}^n,\gamma_n)$. Moreover, $L^{A_G}(\mathbb{R}^n,\gamma_n)$ is the optimal Orlicz range space in (9.6)*.*

In the environment of Lorentz–Zygmund spaces, we have the following results. Note that, according to (2.7), the conditions on the parameters p, q and α in the statement are required to ensure that $L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n)$ is actually an r.i. space.

Theorem 9.6. (i) *Assume that either $p = q = 1$ and $\alpha \geq 0$, or $p \in (1,\infty)$, $q \in [1,\infty]$ and $\alpha \in \mathbb{R}$. Then, there exists a constant $C = C(p,q,\alpha)$ such that*

$$\|u - u_{\gamma_n}\|_{L^{p,q;\alpha+\frac{1}{2}}(\mathbb{R}^n,\gamma_n)} \leq C\|\nabla u\|_{L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n)} \tag{9.7}$$

for every $u \in V^1L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n)$. Moreover, $(L^{p,q;\alpha}(\mathbb{R}^n,\gamma_n), L^{p,q;\alpha+\frac{1}{2}}(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (9.7).

(ii) *Assume that either $q \in [1,\infty)$ and $\alpha + \frac{1}{q} < 0$, or $q = \infty$ and $\alpha \leq 0$. Then, there exists a constant $C = C(q,\alpha)$ such that*

$$\|u - u_{\gamma_n}\|_{L^{\infty,q;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n)} \leq C\|\nabla u\|_{L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n)} \tag{9.8}$$

for every $u \in V^1L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n)$. Moreover, $(L^{\infty,q;\alpha}(\mathbb{R}^n,\gamma_n), L^{\infty,q;\alpha-\frac{1}{2}}(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (9.8).

We have seen in the examples above that the Gaussian Sobolev embeddings provide a *gain* when the domain space lies near $L^1(\mathbb{R}^n,\gamma_n)$, such as

$$V^1L^1(\mathbb{R}^n,\gamma_n) \hookrightarrow L\log L^{\frac{1}{2}}(\mathbb{R}^n,\gamma_n),$$

and a *loss* when the domain space lies near $L^\infty(\mathbb{R}^n,\gamma_n)$, such as

$$V^1L^\infty(\mathbb{R}^n,\gamma_n) \hookrightarrow \exp L^2(\mathbb{R}^n,\gamma_n).$$

We shall now consider the question of existence *self-optimal* spaces in the Gaussian Sobolev inequality (9.2). We first need to introduce some new function spaces. Let $p \in (0,\infty]$ and let $\omega \in \mathcal{M}_+(0,1)$. Then the *classical Lorentz spaces* $\Lambda^p(\omega)(\mathbb{R}^n,\gamma_n)$ and $\Gamma^p(\omega)(\mathbb{R}^n,\gamma_n)$ are defined as the sets of those functions $g \in \mathcal{M}(\mathbb{R}^n,\gamma_n)$ such that the quantities

$$\|g\|_{\Lambda^p(\omega)} = \begin{cases} \left(\int_0^1 g^*(s)^p\omega(s)\,ds\right)^{\frac{1}{p}} & \text{if } p \in (0,\infty) \\ \operatorname{ess\,sup}_{0<s<1} g^*(s)\omega(s) & \text{if } p = \infty, \end{cases}$$

and

$$\|g\|_{\Gamma^p(\omega)} = \begin{cases} \left(\int_0^1 g^{**}(s)^p\omega(s)\,ds\right)^{\frac{1}{p}} & \text{if } p \in (0,\infty) \\ \operatorname{ess\,sup}_{0<s<1} g^{**}(s)\omega(s) & \text{if } p = \infty, \end{cases}$$

respectively, are finite. Clearly, one always has $\Gamma^p(\omega)(\mathbb{R}^n,\gamma_n) \subset \Lambda^p(\omega)(\mathbb{R}^n,\gamma_n)$, and for some p and ω this inclusion may be strict (see [28] and the references therein).

In the case when $p = \infty$, the spaces $\Lambda^\infty(\omega)(\mathbb{R}^n,\gamma_n)$ and $\Gamma^\infty(\omega)(\mathbb{R}^n,\gamma_n)$ often coincide with previously defined Marcinkiewicz spaces. If $\varphi : [0,1) \to [0,\infty)$ has the properties required by a fundamental function, then the space $\Lambda^1(\varphi')(\mathbb{R}^n,\gamma_n)$ has fundamental function φ and it is the smallest such r.i. space. Combined with what we already noticed about a Marcinkiewicz space, this yields for any other r.i. space X with fundamental function $\varphi_X \approx \varphi$, then

$$\Lambda^1(\varphi')(\mathbb{R}^n,\gamma_n) \to X(\mathbb{R}^n,\gamma_n) \to \Gamma^\infty(\varphi)(\mathbb{R}^n,\gamma_n). \tag{9.9}$$

The space $\Lambda^1(\varphi')(\mathbb{R}^n,\gamma_n)$ is then sometimes called the *Lorentz endpoint space*.

We shall find two self-optimal spaces in the Gaussian Sobolev embedding, namely the Lorentz endpoint space and the Marcinkiewicz space whose fundamental function is equivalent to $\varphi_G : (0,1) \to [0,\infty)$, given by

$$\varphi_G(s) = e^{-2\sqrt{1+\log\frac{1}{s}}} \quad \text{for } s \in (0,1). \tag{9.10}$$

Theorem 9.7. *Let φ_G be defined by* (9.10).

(i) *There exists an absolute constant C such that*

$$\|u - u_{\gamma_n}\|_{\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)} \le C\|\nabla u\|_{\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)} \tag{9.11}$$

for every $u \in V^1\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n)$. Moreover, $(\Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n), \Lambda^1(\varphi_G')(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (9.11).

(ii) *There exists an absolute constant C such that*

$$\|u - u_{\gamma_n}\|_{\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)} \le C\|\nabla u\|_{\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)} \tag{9.12}$$

for every $u \in V^1\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n)$. Moreover, $(\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n), \Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (9.12).

Note that, in fact,

$$\Gamma^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n) = \Lambda^\infty(\varphi_G)(\mathbb{R}^n,\gamma_n) \tag{9.13}$$

(up to equivalent norms), as can be easily obtained from an appropriate weighted Hardy inequality.

As a consequence of Theorem 9.7 (ii), one has the following corollary.

Corollary 9.8. *Let A be a Young function fulfilling* (9.3) *and such that $A(t) = e^{\frac{1}{4}\log^2 t}$ for large t. Then, there exists a constant $C = C(A)$ such that*

$$\|u - u_{\gamma_n}\|_{L^A(\mathbb{R}^n,\gamma_n)} \le C\|\nabla u\|_{L^A(\mathbb{R}^n,\gamma_n)} \tag{9.14}$$

for every $u \in V^1L^A(\mathbb{R}^n,\gamma_n)$. Moreover, $(L^A(\mathbb{R}^n,\gamma_n), L^A(\mathbb{R}^n,\gamma_n))$ is an optimal pair in (9.14).

10. Higher-order Sobolev Embeddings and isoperimetric inequalities

Some applications require sharp Sobolev embeddings of *higher order*. As the results show, there is a (perhaps somewhat surprising) significant difference between first-order embeddings and the higher-order ones. To obtain the higher-order reduction theorems is usually a good deal harder problem than its first-order counterpart. For instance, in the case of Euclidean-Sobolev embeddings, the key technical reason for this difference is the lack of a higher-order version of the Pólya–Szegö inequality. The situation is even worse in the case of the Gaussian Sobolev embeddings. The technical difficulties connected are caused by the fact that the corresponding one-dimensional operator becomes a kernel operator when higher-order cases are treated.

We shall focus on the natural question, whether optimal results can be obtained by *iteration* of the first-order embeddings.

It has been known for several decades that, in general, sharpness of the target space can be lost in the process of iteration. This is so for example when limiting cases are treated in context of Orlicz spaces (see e.g. [117]). Even in the simplest possible case, involving just Lebesgue spaces, one can lose optimality. For example, consider the space $W^{2,1}(\Omega)$ in dimension $n = 2$. It is known that this space is embedded into $L^\infty(\Omega)$. However, the best possible Lebesgue space into which the Sobolev space $W^{1,1}(\Omega)$ is embedded into, is $L^2(\Omega)$, so iteration of the first-order embeddings restricted to Lebesgue spaces leads to

$$W^{2,1}(\Omega) \hookrightarrow W^{1,2}(\Omega),$$

but the latter space is no longer embedded into $L^\infty(\Omega)$, so iteration in this case does not provide sharp results. The simple reason for this loss is the fact that the restriction to Lebesgue spaces did not allow us to replace $L^2(\Omega)$ by the Lorentz space $L^{2,1}(\Omega)$, which is sharp in a broad sense, and which would lead to the sharp target via the iteration

$$W^{2,1}(\Omega) \hookrightarrow W^1L^{2,1}(\Omega) \hookrightarrow L^\infty(\Omega).$$

This observation leads us to formulate a natural question. Is the sharpness preserved by the iteration process if, at each step, one considers the optimal target rearrangement-invariant space?

In [44], we developed a general method that under certain restrictions gives an affirmative answer to this question. The method works for every open connected set $\Omega \subset \mathbb{R}^n$, endowed with a non-negative measure ν as long as the isoperimetric inequality is satisfied.

The isoperimetric inequality relative to (Ω, ν) tells us that

$$P_\nu(E, \Omega) \geq I_{\Omega,\nu}(\nu(E)), \tag{10.1}$$

where E is any measurable set $E \subset \Omega$, and $P_\nu(E, \Omega)$ stands for its perimeter in Ω with respect to ν. Moreover, $I_{\Omega,\nu}$ denotes the largest non-decreasing function in $[0, \frac{1}{2}]$ for which (10.1) holds, called the isoperimetric function (or isoperimetric profile) of (Ω, ν), which was introduced in [82].

The isoperimetric function $I_{\Omega,\nu}$ is known *exactly* only in few special instances, e.g. when Ω is an Euclidean ball [84], or agrees with the space $\mathbb{R}^n$ equipped with the Gauss measure [20]. However, the asymptotic behavior of $I_{\Omega,\nu}$ at 0, which in fact is the piece of information relevant for our applications, can be evaluated for various classes of domains, including Euclidean bounded domains whose boundary is locally a graph of a Lipschitz function [84], or, more generally, has a prescribed modulus of continuity [32, 74]; Euclidean John domains, and even s-John domains; the space $\mathbb{R}^n$ equipped with the Gauss measure [20], or with product probability measures which generalize it [10, 9].

Given an r.i. space X on $(\mathbb{R}^n, \gamma_n)$, we set

$$V^mX(\mathbb{R}^n, \gamma_n) = \{u : u \text{ is } m\text{-times weakly differentiable and } \nabla^m u \in X(\mathbb{R}^n, \gamma_n)\}$$

and

$$V^m_\perp X(\mathbb{R}^n, \gamma_n) = \{u \in V^mX(\mathbb{R}^n, \gamma_n) : \textstyle\int_{\mathbb{R}^n} \nabla^k u \, d\gamma_n = 0 \text{ for } k = 0, \dots, m-1\}.$$

The results concerning optimality of function spaces in Sobolev embeddings depend only on a lower bound for the isoperimetric function $I_{\Omega,\nu}$ of (Ω, ν) in terms of some other non-decreasing function $I : [0,1] \to [0,\infty)$; precisely, on the existence of a positive constant c such that

$$I_{\Omega,\nu}(s) \geq cI(cs) \quad \text{for } s \in [0, \tfrac{1}{2}]. \tag{10.2}$$

First, it can be observed that if $I_{\Omega,\nu}(s)$ does not decay at 0 faster than linearly, namely if there exists a positive constant C such that

$$I_{\Omega,\nu}(s) \geq Cs \quad \text{for } s \in [0, \tfrac{1}{2}], \tag{10.3}$$

then any function $u \in V^m X(\Omega,\nu)$ does at least belong to $L^1(\Omega,\nu)$, together with all its derivatives up to the order $m-1$. In the light of (10.3), we can safely assume that

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0. \tag{10.4}$$

We shall now formulate the main general results from [44].

Theorem 10.1. *Assume that* (Ω,ν) *fulfils* (10.2) *for some non-increasing function* I *satisfying* (10.4). *Let* $m \in \mathbb{N}$, *and let* $\|\cdot\|_{X(0,1)}$ *and* $\|\cdot\|_{Y(0,1)}$ *be rearrangement-invariant function norms. If there exists a constant* C_1 *such that*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} ds \right\|_{Y(0,1)} \leq C_1 \, \|f\|_{X(0,1)} \tag{10.5}$$

for every non-negative $f \in X(0,1)$, *then*

$$V^m X(\Omega,\nu) \to Y(\Omega,\nu), \tag{10.6}$$

and there exists a constant C_2 *such that*

$$\|u\|_{Y(\Omega,\nu)} \leq C_2 \, \|\nabla^m u\|_{X(\Omega,\nu)} \tag{10.7}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

It turns out that inequality (10.5) holds for every non-negative $f \in X(0,1)$ if and only if it just holds for every non-negative and non-increasing $f \in X(0,1)$.

A major feature of Theorem 10.1 is the difference occurring in (10.5) between the first-order case ($m = 1$) and the higher-order case ($m > 1$). Indeed, the integral operator appearing in (10.5) when $m = 1$ is just a weighted Hardy-type operator, namely a primitive of f times a weight, whereas, in the higher-order case, a genuine kernel, with a more complicated structure, comes into play. In fact, this seems to be the first known instance where such a kernel operator is needed in a reduction result for Sobolev-type embeddings. Of course, this makes the proof of inequalities of the form (10.5) more challenging, although several contributions on one-dimensional inequalities for kernel operators are available in the literature (see e.g. [75], [60], [87]).

As we shall see, the Sobolev embedding (10.6) (or the Poincaré inequality (10.7)) and inequality (10.5) are actually equivalent in customary families of measure spaces (Ω,ν), and hence, Theorem 4.5 enables us to determine the optimal rearrangement-invariant target spaces in Sobolev embeddings for these measure spaces. Incidentally, let us mention that when $m = 1$, this is the case whenever the geometry of

(Ω,ν) allows the construction of a family of trial functions u in (10.6) or (10.7) characterized by the following properties: the level sets of u are isoperimetric (or almost isoperimetric) in (Ω,ν); $|\nabla u|$ is constant (or almost constant) on the boundary of the level sets of u. If $m>1$, then the latter requirement has to be complemented by requiring that the derivatives of u up to the order m restricted to the boundary of the level sets satisfy certain conditions depending on I.

Such conditions have, however, a technical nature, and it is not worth to state them explicitly. In fact, heuristically speaking, properties (10.5), (10.7) and (10.6) turn out to be equivalent for every $m\ge 1$ on the same measure spaces (Ω,ν) as for $m=1$. Such an equivalence certainly holds in any customary, non-pathological situation, including the three frameworks to which our results will be applied, namely John domains, Euclidean domains from Maz'ya classes and product probability spaces in $\mathbb{R}^n$ extending the Gauss space. In all these cases, Theorem 4.5 provides us with a characterization of optimal arbitrary-order rearrangement-invariant target spaces.

Now we are in a position to characterize the space which is the optimal rearrangement-invariant target space in the Sobolev embedding (10.6). Such an optimal space is the one associated with the rearrangement-invariant function norm $\|\cdot\|_{X_{m,I}(0,1)}$, whose associate norm is defined as

$$\|f\|_{X'_{m,I}(0,1)}=\left\|\frac{1}{I(s)}\int_0^s\left(\int_t^s\frac{dr}{I(r)}\right)^{m-1}f^*(t)dt\right\|_{X'(0,1)} \tag{10.8}$$

for $f\in\mathcal{M}_+(0,1)$.

Theorem 10.2. *Assume that (Ω,ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 10.1. Then the functional $\|\cdot\|_{X'_{m,I}(0,1)}$, given by* (10.8)*, is equivalent to a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,I}(0,1)}$ satisfies*

$$V^mX(\Omega,\nu)\to X_{m,I}(\Omega,\nu), \tag{10.9}$$

and there exists a constant C such that

$$\|u\|_{X_{m,I}(\Omega,\nu)}\le C\,\|\nabla^m u\|_{X(\Omega,\nu)} \tag{10.10}$$

for every $u\in V^m_\perp X(\Omega,\nu)$.

Moreover, if (Ω,ν) is such that (10.6)*, or equivalently* (10.7)*, implies* (10.5)*, and hence* (10.5)*,* (10.6) *and* (10.7) *are equivalent, then the function norm $\|\cdot\|_{X_{m,I}(0,1)}$ is optimal in* (10.9) *and* (10.10) *among all rearrangement-invariant norms.*

An important special case of Theorems 10.1 and 10.2 is enucleated in the following corollary.

Corollary 10.3. *Assume that (Ω,ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 10.1. If*

$$\left\|\frac{1}{I(s)}\left(\int_0^s\frac{dr}{I(r)}\right)^{m-1}\right\|_{X'(0,1)}<\infty\,, \tag{10.11}$$

then

$$V^mX(\Omega,\nu)\to L^\infty(\Omega,\nu), \tag{10.12}$$

and there exists a constant C such that

$$\|u\|_{L^\infty(\Omega,\nu)} \le C \|\nabla^m u\|_{X(\Omega,\nu)} \tag{10.13}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

Moreover, if (Ω,ν) is such that (10.6)*, or equivalently* (10.7)*, implies* (10.5)*, and hence* (10.5), (10.6) *and* (10.7) *are equivalent, then* (10.11) *is necessary for* (10.12) *or* (10.13) *to hold.*

If (Ω,ν) is such that (10.6), or equivalently (10.7), implies (10.5), and hence (10.5), (10.6) and (10.7) are equivalent, then (8.24) cannot hold, whatever $\|\cdot\|_{X(0,1)}$ is, if I decays so fast at 0 that

$$\int_0 \frac{dr}{I(r)} = \infty.$$

We shall now point out the preservation of optimality in targets among all rearrangement-invariant spaces under iteration of Sobolev embeddings of arbitrary order.

Theorem 10.4. *Assume that (Ω,ν), I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 10.1. Let $k, h \in \mathbb{N}$. Then*

$$(X_{k,I})_{h,I}(\Omega,\nu) = X_{k+h,I}(\Omega,\nu), \tag{10.14}$$

up to equivalent norms.

In many instances in practice, the function I satisfies the estimate

$$\int_0^s \frac{dr}{I(r)} \approx \frac{s}{I(s)} \quad \text{for } s \in (0,1). \tag{10.15}$$

It is useful to treat these cases separately because if the function I satisfies (10.15), then the results of Theorems 10.1, 10.2 and 10.4 can be considerably simplified. For example, the reduction theorem then reads as follows.

Corollary 10.5. *Let (Ω,ν), m, I, $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be as in Theorem 10.1. Assume, in addition, that I fulfils* (10.15)*. If there exists a constant C_1 such that*

$$\left\| \int_t^1 f(s) \frac{s^{m-1}}{I(s)^m}\, ds \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{10.16}$$

for every non-negative $f \in X(0,1)$, then

$$V^m X(\Omega,\nu) \to Y(\Omega,\nu), \tag{10.17}$$

and there exists a constant C_2 such that

$$\|u\|_{Y(\Omega,\nu)} \le C_2 \|\nabla^m u\|_{X(\Omega,\nu)} \tag{10.18}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

The next corollary tells us that, under the extra condition (10.15), the optimal rearrangement-invariant target space takes a simplified form. Namely, it can be equivalently defined via the rearrangement-invariant function norm $\|\cdot\|_{X^\sharp_{m,I}(0,1)}$ obeying

$$\|f\|_{(X^\sharp_{m,I})'(0,1)} = \left\| \frac{t^{m-1}}{I(t)^m} \int_0^t f^*(s)\,ds \right\|_{X'(0,1)} \tag{10.19}$$

for every $f \in \mathcal{M}_+(0,1)$.

Corollary 10.6. *Assume that (Ω,ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Corollary 10.5. Then the functional $\|\cdot\|_{(X^\sharp_{m,I})'(0,1)}$, given by* (10.19)*, is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X^\sharp_{m,I}(0,1)}$ satisfies*

$$V^m X(\Omega,\nu) \to X^\sharp_{m,I}(\Omega,\nu), \tag{10.20}$$

and there exists a constant C such that

$$\|u\|_{X^\sharp_{m,I}(\Omega,\nu)} \le C\,\|\nabla^m u\|_{X(\Omega,\nu)} \tag{10.21}$$

for every $u \in V^m_\perp X(\Omega,\nu)$. Moreover, if (Ω,ν) is such that the validity of (10.17)*, or equivalently* (10.18)*, implies* (10.16)*, and hence* (10.16)*,* (10.17) *and* (10.18) *are equivalent, then the function norm $\|\cdot\|_{X^\sharp_{m,I}(0,1)}$ is optimal in* (10.20) *and* (10.21) *among all rearrangement-invariant norms.*

Let us now deal with specific situations separately.

We say that a bounded open set Ω in $\mathbb{R}^n$ is called a *John domain* if there exist a constant $c \in (0,1)$ and a point $x_0 \in \Omega$ such that for every $x \in \Omega$ there exists a rectifiable curve $\varpi : [0,l] \to \Omega$, parameterized by the arc length and such that $\varpi(0) = x$, $\varpi(l) = x_0$, and

$$\operatorname{dist}(\varpi(r), \partial\Omega) \ge cr \qquad \text{for } r \in [0,l].$$

The class of John domains includes other more classical families of domains, such as Lipschitz domains, and domains with the cone property. The John domains arise in connection with the study of holomorphic dynamical systems and quasiconformal mappings. John domains are known to support a first-order Sobolev inequality with the same exponents as in the standard Sobolev inequality [20, 58, 71]. In fact, being a John domain is a necessary condition for such a Sobolev inequality to hold in the class of two-dimensional simply connected open sets, and in quite general classes of higher-dimensional domains [26]. The isoperimetric function $I_\Omega(s)$ of any John domain is known to satisfy

$$I_\Omega(s) \approx s^{\frac{1}{n'}} \tag{10.22}$$

near 0. As a consequence of (10.22), one can show that the first-order Sobolev embedding

$$V^1 X(\Omega) \to Y(\Omega) \tag{10.23}$$

holds if and only if

$$\left\| \int_t^1 f(s) s^{-1+\frac{1}{n}}\,ds \right\|_{Y(0,1)} \le C\,\|f\|_{X(0,1)} \tag{10.24}$$

for some constant C, and for every non-negative $f \in X(0,1)$, which slightly generalizes Theorem 4.1 (from Lipschitz domains to John ones).

Theorem 10.7. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $m \in \mathbb{N}$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following assertions are equivalent.*

(i) *The Hardy type inequality*

$$\left\| \int_t^1 f(s) s^{-1+\frac{m}{n}}\, ds \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)} \tag{10.25}$$

holds for some constant C_1, and for every non-negative $f \in X(0,1)$.

(ii) *The Sobolev embedding*

$$V^m X(\Omega) \to Y(\Omega) \tag{10.26}$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\Omega)} \leq C_2 \|\nabla^m u\|_{X(\Omega)} \tag{10.27}$$

holds for some constant C_2 and every $u \in V_\perp^m X(\Omega)$.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$ and $m \in \mathbb{N}$, we define $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ as the rearrangement-invariant function norm, whose associate function norm is given by

$$\|f\|_{X'_{m,\mathrm{John}}(0,1)} = \left\| s^{-1+\frac{m}{n}} \int_0^s f^*(r) dr \right\|_{X'(0,1)} \tag{10.28}$$

for $f \in \mathcal{M}_+(0,1)$. The function norm $\|\cdot\|_{X(0,1)}$ is optimal, as a target, for Sobolev embeddings of $V^m X(\Omega)$.

Theorem 10.8. *Let n, m, Ω and $\|\cdot\|_{X(0,1)}$ be as in Theorem 10.7. Then the functional $\|\cdot\|_{X'_{m,\mathrm{John}}(0,1)}$, given by (10.28), is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ satisfies*

$$V^m X(\Omega) \to X_{m,\mathrm{John}}(\Omega), \tag{10.29}$$

and

$$\|u\|_{X_{m,\mathrm{John}}(\Omega)} \leq C \|\nabla^m u\|_{X(\Omega)} \tag{10.30}$$

for some constant C and every $u \in V_\perp^m X(\Omega, \nu)$. Moreover, the function norm $\|\cdot\|_{X_{m,\mathrm{John}}(0,1)}$ is optimal in (10.29) and (10.30) among all rearrangement-invariant norms.

Our next set of instances will be *Maz'ya classes* of domains. Given $\alpha \in [\frac{1}{n'}, 1]$, we denote by $\mathcal{J}_\alpha$ the *Maz'ya class* of all Euclidean domains Ω satisfying (10.2), with $I(s) = s^\alpha$ for $s \in [0, \frac{1}{2}]$, namely domains Ω in $\mathbb{R}^n$ such that

$$I_\Omega(s) \geq C s^\alpha \quad \text{for } s \in [0, \tfrac{1}{2}], \tag{10.31}$$

for some positive constant C. Thanks to (10.22), any John domain belongs to the class $\mathcal{J}_{\frac{1}{n'}}$.

The reduction theorem in the class $\mathcal{J}_\alpha$ takes the following form.

Theorem 10.9. *Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, and $\alpha \in [\frac{1}{n'}, 1]$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Assume that either $\alpha \in [\frac{1}{n'}, 1)$ and there exists a constant C_1 such that*

$$\left\| \int_t^1 f(s) s^{-1+m(1-\alpha)}\, ds \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)} \tag{10.32}$$

for every non-negative $f \in X(0,1)$, or $\alpha = 1$ and there exists a constant C_1 such that

$$\left\| \int_t^1 f(s) \frac{1}{s} \left(\log \frac{s}{t} \right)^{m-1} ds \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)} \tag{10.33}$$

for every non-negative $f \in X(0,1)$. Then the Sobolev embedding

$$V^m X(\Omega) \to Y(\Omega) \tag{10.34}$$

holds for every $\Omega \in \mathcal{J}_\alpha$ and, equivalently, the Poincaré inequality

$$\|u\|_{Y(\Omega)} \leq C_2 \|\nabla^m u\|_{X(\Omega)} \tag{10.35}$$

holds for every $\Omega \in \mathcal{J}_\alpha$, for some constant C_2 and every $u \in V_\perp^m X(\Omega)$.

Conversely, if the Sobolev embedding (10.34)*, or, equivalently, the Poincaré inequality* (10.35)*, holds for every $\Omega \in \mathcal{J}_\alpha$, then either inequality* (10.32)*, or* (10.33) *holds, according to whether $\alpha \in [\frac{1}{n'}, 1)$ or $\alpha = 1$.*

A major consequence of Theorem 10.9 is the identification of the optimal rearrangement-invariant target space $Y(\Omega)$ associated with a given domain $X(\Omega)$ in embedding (10.34) as Ω is allowed to range among all domains in the class $\mathcal{J}_\alpha$. This is the content of the next result. The rearrangement-invariant function norm yielding such an optimal space will be denoted by $\|\cdot\|_{X_{m,\alpha}(0,1)}$. Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, $m \in \mathbb{N}$, and $\alpha \in [\frac{1}{n'}, 1]$, it is characterized through its associate function norm defined by

$$\|f\|_{X'_{m,\alpha}(0,1)} = \begin{cases} \left\| s^{-1+m(1-\alpha)} \int_0^s f^*(r) dr \right\|_{X'(0,1)} & \text{if } \alpha \in [\frac{1}{n'}, 1), \\ \\ \left\| \frac{1}{s} \int_0^s \left(\log \frac{s}{r} \right)^{m-1} f^*(r) dr \right\|_{X'(0,1)} & \text{if } \alpha = 1, \end{cases} \tag{10.36}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 10.10. *Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, α and $\|\cdot\|_{X(0,1)}$ be as in Theorem 10.9. Then the functional $\|\cdot\|_{X'_{m,\alpha}(0,1)}$, given by* (10.36)*, is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,\alpha}(0,1)}$ satisfies*

$$V^m X(\Omega) \to X_{m,\alpha}(\Omega) \tag{10.37}$$

for every $\Omega \in \mathcal{J}_\alpha$, and

$$\|u\|_{X_{m,\alpha}(\Omega)} \leq C \|\nabla^m u\|_{X(\Omega)} \tag{10.38}$$

for every $\Omega \in \mathcal{J}_\alpha$, for some constant C and every $u \in V_\perp^m X(\Omega)$. Moreover, the function norm $\|\cdot\|_{X_{m,\alpha}(0,1)}$ is optimal in (10.37) *and* (10.38) *among all rearrangement-invariant norms, as Ω ranges in $\mathcal{J}_\alpha$.*

Theorem 10.10 is a straightforward consequence of Theorem 10.9, and either Corollary 10.6 or Theorem 4.5, according to whether $\alpha \in [\frac{1}{n'}, 1)$ or $\alpha = 1$.

We shall now apply the general results to some concrete function spaces. We shall mainly focus on Lebesgue, Lorentz and Orlicz spaces.

Theorem 10.11. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_\alpha$ for some $\alpha \in [\frac{1}{n'}, 1)$. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then*

(10.39)

$$V^m L^p(\Omega) \to \begin{cases} L^{\frac{p}{1-mp(1-\alpha)}}(\Omega) & \text{if } m(1-\alpha) < 1 \text{ and } 1 \leq p < \frac{1}{m(1-\alpha)}, \\ L^r(\Omega) & \text{for any } r \in [1,\infty), \text{ if } m(1-\alpha) < 1 \text{ and } p = \frac{1}{m(1-\alpha)}, \\ L^\infty(\Omega) & \text{otherwise.} \end{cases}$$

Moreover, in the first and the third cases, the target spaces in (10.39) *are optimal among all Lebesgue spaces, as Ω ranges in $\mathcal{J}_\alpha$.*

Although the target spaces in (10.39) cannot be improved in the class of Lebesgue spaces, the conclusions of (10.39) can be strengthened if more general rearrangement-invariant spaces are employed. Such a strengthening can be obtained as a special case of a Sobolev embedding for Lorentz spaces which reads as follows.

Theorem 10.12. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_\alpha$ for some $\alpha \in [\frac{1}{n'}, 1)$. Let $m \in \mathbb{N}$ and $p, q \in [1, \infty]$. Assume that one of the conditions in* (2.4) *holds. Then*

(10.40)

$$V^m L^{p,q}(\Omega) \to \begin{cases} L^{\frac{p}{1-mp(1-\alpha)},q}(\Omega) & \text{if } m(1-\alpha) < 1 \text{ and } 1 \leq p < \frac{1}{m(1-\alpha)}, \\ L^{\infty,q;-1}(\Omega) & \text{if } m(1-\alpha) < 1,\ p = \frac{1}{m(1-\alpha)} \text{ and } q > 1, \\ L^\infty(\Omega) & \text{otherwise,} \end{cases}$$

Moreover, the target spaces in (10.40) *are optimal among all rearrangement-invariant spaces, as Ω ranges in $\mathcal{J}_\alpha$.*

The particular choice of parameters $p = q$, $1 \leq p < \frac{1}{m(1-\alpha)}$ in Theorem 10.12 shows that

$$V^m L^p(\Omega) \hookrightarrow L^{\frac{p}{1-mp(1-\alpha)},p}(\Omega).$$

This is a non-trivial strengthening of the first embedding in (10.39), since

$$L^{\frac{p}{1-mp(1-\alpha)},p}(\Omega) \subsetneq L^{\frac{p}{1-mp(1-\alpha)}}.$$

Likewise, the choice $m(1-\alpha) < 1$ and $p = q = \frac{1}{m(1-\alpha)}$ shows that also the second embedding in (10.39) can be in fact essentially improved by

$$V^m L^p(\Omega) \hookrightarrow L^{\infty,p;-1}(\Omega).$$

Assume now that $\alpha = 1$. The embedding theorem in Lebesgue spaces takes the following form.

Theorem 10.13. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\Omega \in \mathcal{J}_1$. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then*

$$V^m L^p(\Omega) \to \begin{cases} L^p(\Omega) & \text{if } 1 \leq p < \infty, \\ L^r(\Omega) & \text{for any } r \in [1,\infty), \text{ if } p = \infty. \end{cases} \tag{10.41}$$

The target spaces are optimal in (10.41) *among all Lebesgue spaces, as* Ω *ranges in* $\mathcal{J}_1$.

Optimal embeddings for Lorentz-Sobolev spaces are provided in the next theorem.

Theorem 10.14. *Let* $n \in \mathbb{N}$, $n \geq 2$, *and let* $\Omega \in \mathcal{J}_1$. *Let* $m \in \mathbb{N}$ *and* $p, q \in [1, \infty]$. *Assume that one of the conditions in* (2.4) *holds. Then*

$$V^m L^{p,q}(\Omega) \to \begin{cases} L^{p,q}(\Omega) & \text{if } 1 \leq p < \infty, \\ \exp L^{\frac{1}{m}}(\Omega) & \text{if } p = q = \infty. \end{cases} \tag{10.42}$$

The target spaces are optimal in (10.42) *among all rearrangement-invariant spaces, as* Ω *ranges in* $\mathcal{J}_1$.

Our last application in this section concerns Orlicz-Sobolev spaces. Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$, and let A be a Young function. We may assume, without loss of generality, that

$$\int_0 \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt < \infty. \tag{10.43}$$

Indeed, the function A can be modified near 0, if necessary, in such a way that (10.43) is fulfilled, on leaving the space $V^m L^A(\Omega)$ unchanged (up to equivalent norms).

If $m < \frac{1}{1-\alpha}$ and the integral

$$\int^\infty \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt \tag{10.44}$$

diverges, we define the function $H_{m,\alpha} : [0, \infty) \to [0, \infty)$ as

$$H_{m,\alpha}(s) = \left(\int_0^s \left(\frac{t}{A(t)}\right)^{\frac{m(1-\alpha)}{1-m(1-\alpha)}} dt\right)^{1-m(1-\alpha)} \qquad \text{for } s \geq 0, \tag{10.45}$$

and the Young function $A_{m,\alpha}$ as

$$A_{m,\alpha}(t) = A(H_{m,\alpha}^{-1}(t)) \qquad \text{for } t \geq 0. \tag{10.46}$$

Theorem 10.15. *Assume that* $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$ *and* $\Omega \in \mathcal{J}_\alpha$. *Let* A *be a Young function fulfilling* (10.43). *Then*

$$V^m L^A(\Omega) \to \begin{cases} L^{A_{m,\alpha}}(\Omega) & \text{if } m < \frac{1}{1-\alpha}, \text{ and the integral (10.44) diverges,} \\ L^\infty(\Omega) & \text{if either } m \geq \frac{1}{1-\alpha}, \text{ or } m < \frac{1}{1-\alpha} \\ & \text{and the integral (10.44) converges.} \end{cases} \tag{10.47}$$

Moreover, the target spaces in (10.47) *are optimal among all Orlicz spaces, as* Ω *ranges in* $\mathcal{J}_\alpha$.

Example 10.16. Consider the case when

$A(t) \approx t^p (\log t)^\beta$ near infinity, where either $p > 1$ and $\beta \in R$, or $p = 1$ and $\beta \geq 0$.

Hence, $L^A(\Omega) = L^p\log L^\beta(\Omega)$. An application of Theorem 10.15 tells us that (10.48)

$$V^m L^p\log^\beta L(\Omega) \to \begin{cases} L^{\frac{p}{1-pm(1-\alpha)}}\log^{\frac{\beta}{1-pm(1-\alpha)}} L(\Omega) & \text{if } mp(1-\alpha) < 1, \\ \exp L^{\frac{1}{1-(1+\beta)m(1-\alpha)}}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta < \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ \exp\exp L^{\frac{1}{1-m(1-\alpha)}}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta = \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ L^\infty(\Omega) & \text{if either } mp(1-\alpha) > 1, \\ & \text{or } mp(1-\alpha) = 1 \text{ and } \beta > \frac{1-m(1-\alpha)}{m(1-\alpha)}. \end{cases}$$

Moreover, the target spaces in (10.48) are optimal among all Orlicz spaces, as Ω ranges in $\mathcal{J}_\alpha$. The first three embeddings in (10.48) can be improved on allowing more general rearrangement-invariant target spaces. Indeed, we have that (10.49)

$$V^m L^p\log^\beta L(\Omega) \to \begin{cases} L^{\frac{p}{1-pm(1-\alpha)},p;\frac{\beta}{p}}(\Omega) & \text{if } mp(1-\alpha) < 1, \\ L^{\infty,\frac{1}{m(1-\alpha)};m(1-\alpha)\beta-1}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta < \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ L^{\infty,\frac{1}{m(1-\alpha)};-m(1-\alpha),-1}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta = \frac{1-m(1-\alpha)}{m(1-\alpha)}, \end{cases}$$

the targets being optimal among all rearrangement-invariant spaces in (10.49) as Ω ranges among all domains in $\mathcal{J}_\alpha$.

Our final set of examples will concern the *product probability spaces.*

The class of product probability measures in $\mathbb{R}^n$, $n \geq 1$, arises in connection with the study of generalized hypercontractivity theory and integrability properties of the associated heat semigroups. The isoperimetric problem in the corresponding probability spaces was studied in [10], see also [9, 18, 77, 78].

Assume that $\Phi : [0,\infty) \to [0,\infty)$ is a strictly increasing, twice continuously differentiable convex function in $[0,\infty)$ such that $\sqrt{\Phi}$ is concave, and $\Phi(0) = 0$. Let μ_Φ be the probability measure on $\mathbb{R}$ given by

$$d\mu_\Phi(x) = c_\Phi e^{-\Phi(|x|)}\,dx, \tag{10.50}$$

where c_Φ is a constant chosen in such a way that $\mu_\Phi(\mathbb{R}) = 1$. The product measure $\mu_{\Phi,n}$ on $\mathbb{R}^n$, $n \geq 1$, generated by μ_Φ, is then defined as

$$\mu_{\Phi,n} = \underbrace{\mu_\Phi \times \cdots \times \mu_\Phi}_{n-\text{times}}. \tag{10.51}$$

Clearly, $\mu_{\Phi,1} = \mu_\Phi$, and $(\mathbb{R}^n, \mu_{\Phi,n})$ is a probability space for every $n \in \mathbb{N}$. The main example of a measure μ_Φ is obtained by taking

$$\Phi(t) = \tfrac{1}{2}t^2.$$

This choice yields $\mu_{\Phi,n} = \gamma_n$, the Gauss measure which obeys

$$d\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}\,dx. \tag{10.52}$$

More generally, the *Boltzmann measures* $\gamma_{n,\beta}$ are associated with

$$\Phi(t) = \tfrac{1}{\beta}t^\beta$$

for some $\beta \in [1,2]$.

Let $H : \mathbb{R} \to (0,1)$ be defined as

$$H(t) = \int_t^\infty c_\Phi e^{-\Phi(|r|)}\, dr \quad \text{for } t \in \mathbb{R}, \tag{10.53}$$

and let $F_\Phi : [0,1] \to [0,\infty)$ be given by

$$F_\Phi(s) = c_\Phi e^{-\Phi(|H^{-1}(s)|)} \quad \text{for } s \in (0,1), \quad \text{and} \quad F_\Phi(0) = F_\Phi(1) = 0. \tag{10.54}$$

Since μ_Φ is a probability measure and $\mu_{\Phi,n}$ is defined by (10.51), it is easily seen that, for each $i = 1, \dots, n$,

$$\mu_{\Phi,n}(\{(x_1, \dots, x_n) : x_i > t\}) = H(t) \quad \text{for } t \in \mathbb{R}, \tag{10.55}$$

and

$$P_{\mu_{\Phi,n}}(\{(x_1, \dots, x_n) : x_i > t\}, \mathbb{R}^n) = c_\Phi e^{-\Phi(|t|)} = -H'(t) \quad \text{for } t \in \mathbb{R}. \tag{10.56}$$

Hence, $F_\Phi(s)$ agrees with the perimeter of any half-space of the form $\{x_i > t\}$, whose measure is s.

Next, define $L_\Phi : [0,1] \to [0,\infty)$ as

$$L_\Phi(s) = s\Phi'\big(\Phi^{-1}(\log\left(\tfrac{2}{s}\right))\big) \qquad \text{for } s \in (0,1], \quad \text{and} \quad L_\Phi(0) = 0. \tag{10.57}$$

Then the isoperimetric function of $(\mathbb{R}^n, \mu_{\Phi,n})$ satisfies

$$I_{(\mathbb{R}^n, \mu_{\Phi,n})}(s) \approx F_\Phi(s) \approx L_\Phi(s) \quad \text{for } s \in [0, \tfrac{1}{2}] \tag{10.58}$$

(see [10, Proposition 13 and Theorem 15]). Furthermore, half-spaces, whose boundary is orthogonal to a coordinate axis, are “approximate solutions" to the isoperimetric problem in $(\mathbb{R}^n, \mu_{\Phi,n})$ in the sense that there exist constants C_1 and C_2, depending on n, such that, for every $s \in (0,1)$, any such half-space with measure is s satisfies

$$C_1 P_{\mu_{\Phi,n}}(V, \mathbb{R}^n) \le I_{(\mathbb{R}^n, \mu_{\Phi,n})}(s) \le C_2 P_{\mu_{\Phi,n}}(V, \mathbb{R}^n).$$

In the special case when $\mu_{\Phi,n} = \gamma_n$, the Gauss measure, equation (10.58) yields

$$I_{(\mathbb{R}^n, \gamma_n)}(s) \approx s\big(\log \tfrac{2}{s}\big)^{\frac{1}{2}} \quad \text{for } s \in (0, \tfrac{1}{2}]. \tag{10.59}$$

Moreover, any half-space is, in fact, an exact minimizer in the isoperimetric inequality [20, 110].

The reduction theorem for Sobolev embeddings in product probability spaces reads as follows.

Theorem 10.17. *Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, let $\mu_{\Phi,n}$ be the probability measure defined by (10.51), and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The inequality*

$$\left\| \int_s^1 f(r) \frac{\left(\Phi^{-1}\left(\log \frac{2}{s}\right) - \Phi^{-1}\left(\log \frac{2}{r}\right)\right)^{m-1}}{r\Phi'\left(\Phi^{-1}\left(\log \frac{2}{r}\right)\right)}\, dr \right\|_{Y(0,1)} \le C_1 \|f\|_{X(0,1)} \tag{10.60}$$

holds for some constant C_1, and for every non-negative $f \in X(0,1)$.

(ii) *The embedding*

$$V^m X(\mathbb{R}^n, \mu_{\Phi,n}) \to Y(\mathbb{R}^n, \mu_{\Phi,n}) \tag{10.61}$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\mu_{\Phi,n})} \le C_2 \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_{\Phi,n})} \tag{10.62}$$

holds for some constant C_2, and for every $u \in V_\perp^m X(\mathbb{R}^n,\mu_{\Phi,n})$.

The rearrangement-invariant function norm $\|\cdot\|_{X_{m,\Phi}(0,1)}$ which yields the optimal rearrangement-invariant target space $Y(\mathbb{R}^n,\mu_{\Phi,n})$ in embedding (10.61) is defined as follows. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm, and let $n, m \in \mathbb{N}$. Then $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is the rearrangement-invariant function norm whose associate function norm is given by

$$\|f\|_{X'_{m,\Phi}(0,1)} = \left\| \int_0^r f^*(s) \frac{\left(\Phi^{-1}\left(\log\frac{2}{s}\right) - \Phi^{-1}\left(\log\frac{2}{r}\right)\right)^{m-1}}{r\Phi'\left(\Phi^{-1}\left(\log\frac{2}{r}\right)\right)}\, ds \right\|_{X'(0,1)} \tag{10.63}$$

for $f \in \mathcal{M}_+(0,1)$.

The reduction theorem takes a simpler form in the cause of Gaussian measure.

Theorem 10.18. *Let $X(\mathbb{R}^n,\gamma_n)$ and $Y(\mathbb{R}^n,\gamma_n)$ be r.i. spaces, and let $m \ge 1$. There exists a constant C_1 such that*

$$\|u\|_{Y(\mathbb{R}^n,\gamma_n)} \le C_1 \|\nabla^m u\|_{X(\mathbb{R}^n,\gamma_n)}$$

for every $u \in V_\perp^m X(\mathbb{R}^n,\gamma_n)$ if and only if there exists a constant C_2 such that

$$\left\| \frac{1}{\left(1+\log\frac{1}{s}\right)^{\frac{m-1}{2}}} \int_s^1 f(r) \frac{\left(\log\frac{r}{s}\right)^{m-1}}{r\left(1+\log\frac{1}{r}\right)^{1/2}}\, dr \right\|_{Y(0,1)} \le C_2 \|f\|_{X(0,1)}$$

for every $f \in X(0,1)$.

Following the standard line of argument, we can now characterize the optimal range space when the target space is given.

Theorem 10.19. *Let $X(\mathbb{R}^n,\gamma_n)$ be an r.i. space, and let $X_m(\mathbb{R}^n,\gamma_n)$ be the r.i. space whose associate norm is given by*

$$\|u\|_{(X_m)'(\mathbb{R}^n,\gamma_n)} = \left\| \frac{1}{r\left(1+\log\frac{1}{r}\right)^{1/2}} \int_0^r u^*(s) \frac{\left(\log\frac{r}{s}\right)^{m-1}}{\left(1+\log\frac{1}{s}\right)^{\frac{m-1}{2}}}\, ds \right\|_{X'(0,1)}$$

for any $u \in \mathcal{M}(\mathbb{R}^n)$. Then there exists a constant C such that

$$\|u\|_{X_m(\mathbb{R}^n,\gamma_n)} \le C \|\nabla^m u\|_{X(\mathbb{R}^n,\gamma_n)}$$

for every $u \in V_\perp^m X(\mathbb{R}^n,\gamma_n)$. Moreover, the space $X_m(\mathbb{R}^n,\gamma_n)$ is optimal among all r.i. spaces.

We finish this section with an application of the results of this section to the particular case when $\mu_{\Phi,n}$ is a Boltzmann measure, and the norms are of Lorentz–Zygmund type.

Theorem 10.20. *Let $n, m \in \mathbb{N}$, let $\beta \in [1,2]$ and let $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in* (2.7) *is satisfied. Then*

$$V^m L^{p,q;\alpha}(\mathbb{R}^n, \gamma_{n,\beta}) \to \begin{cases} L^{p,q;\alpha+\frac{m(\beta-1)}{\beta}}(\mathbb{R}^n, \gamma_{n,\beta}) & \text{if} \quad p < \infty; \\ L^{\infty,q;\alpha-\frac{m}{\beta}}(\mathbb{R}^n, \gamma_{n,\beta}) & \text{if} \quad p = \infty. \end{cases} \tag{10.64}$$

Moreover, in both cases, the target space is optimal among all rearrangement-invariant spaces.

When $\beta = 2$, Theorem 10.20 yields the following sharp Sobolev type embeddings in Gauss space.

Theorem 10.21. *Let $n, m \in \mathbb{N}$, and let $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in* (2.7) *is satisfied. Then*

$$V^m L^{p,q;\alpha}(\mathbb{R}^n, \gamma_n) \to \begin{cases} L^{p,q;\alpha+\frac{m}{2}}(\mathbb{R}^n, \gamma_n) & \text{if} \quad p < \infty; \\ L^{\infty,q;\alpha-\frac{m}{2}}(\mathbb{R}^n, \gamma_n) & \text{if} \quad p = \infty. \end{cases}$$

Moreover, in both cases, the target space is optimal among all rearrangement-invariant spaces.

A further specialization of the indices p, q, α appearing in Theorem 10.21 leads to the following basic embeddings.

Corollary 10.22. *Let $n, m \in \mathbb{N}$.*

(i) *Assume that $p \in [1,\infty)$. Then*

$$V^m L^p(\mathbb{R}^n, \gamma_n) \to L^p(\log L)^{\frac{mp}{2}}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

(ii) *Assume that $\gamma > 0$. Then*

$$V^m \exp L^{\gamma}(\mathbb{R}^n, \gamma_n) \to \exp L^{\frac{2\gamma}{2+m\gamma}}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

(iii)

$$V^m L^{\infty}(\mathbb{R}^n, \gamma_n) \to \exp L^{\frac{2}{m}}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

11. Oscillation of Sobolev functions

We have seen various forms of embeddings of the basic Sobolev space built upon Lebesgue space, that is, $W^{1,p}(\Omega)$, into other function spaces in the *sublimiting* case (see (2.1)) and also in the *limiting* case (see (2.2), (2.3) and (2.15)). The *superlimiting* case $p > n$ is of course also of interest. However, when the target space of such embedding is restricted to rearrangement-invariant spaces, then the results are not so interesting, since the smallest rearrangement-invariant space is $L^{\infty}(\Omega)$ (as long as Ω is of finite measure), while usually a lot more can be said about the properties of functions from superlimiting Sobolev spaces, for example their oscillation or smoothness properties.

Smoothness of functions is usually measured by their belongness to certain types of Hölder spaces, for pointwise oscillation the Morrey spaces constitute the appropriate target, and for the mean oscillation their role is taken by Campanato spaces. In this section we shall investigate Sobolev embeddings into such function spaces.

The space BMO of functions having *bounded mean oscillation*, introduced by John and Nirenberg ([64]), has proved to be particularly useful in various areas of analysis, especially harmonic analysis (see [112] or [109, Chapter 4] and the references given there) and interpolation theory (see [15, Chapter 5]), as an appropriate substitute for L^∞ when L^∞ does not work. The related space of functions with *vanishing mean oscillation*, called VMO, is also of interest and has a number of applications in the theory of partial differential equations, see e.g. [31, 30, 21].

Let Q be a cube in $\mathbb{R}^n$. The space $\mathrm{BMO}(Q)$ is defined as the class of real-valued integrable functions on Q such that

$$\|f\|_{*,Q} = \sup_{Q'\subset Q} \frac{1}{|Q'|} \int_{Q'} |f(x) - f_{Q'}|\,dx < \infty,$$

where the supremum is extended over all subcubes Q' of Q, and $f_{Q'} = |Q'|^{-1}\int_{Q'} f$. Let us recall that BMO is not a Banach space, although it can be turned into one by introducing the norm

$$\|f\|_{\mathrm{BMO}(Q)} = \|f\|_{*,Q} + \|f\|_{L^1(Q)}.$$

We say that a function $f : Q \to \mathbb{R}$ belongs to $\mathrm{VMO}(Q)$, the space of functions with *vanishing mean oscillation*, if $\lim_{s\to 0_+} \varrho_f(s) = 0$, where

$$\varrho_f(s) = \sup_{|Q'|\le s} \frac{1}{|Q'|} \int_{Q'} |f(x) - f_{Q'}|\,dx.$$

The following relations hold: $L^\infty \subsetneq \mathrm{BMO}$, $\mathrm{VMO} \subsetneq \mathrm{BMO}$, $L^\infty \not\subset \mathrm{VMO}$, and $\mathrm{VMO} \not\subset L^\infty$ (the non-equalities and non-inclusions can be demonstrated for example with the functions $\log|x|$, $\log|x|$, $\sin(\log|x|)$, and $(\log|x|)^{\frac{1}{2}}$, respectively).

The following characterization of Sobolev embedding into BMO was established in [39].

Theorem 11.1. *Let $X(Q)$ be a rearrangement-invariant space. Then the Sobolev embedding*

$$W^1X(Q) \hookrightarrow \mathrm{BMO}(Q)$$

holds if and only if

$$\sup_{0<t<1} \tfrac{1}{t} \left\| s^{\frac{1}{n}} \chi_{(0,t)}(s) \right\|_{X'(0,1)} < \infty.$$

As a corollary, we find that the optimal domain for the Sobolev embedding in which BMO is a fixed target, is the Lorentz space $L^{n,\infty}(Q)$.

Corollary 11.2. *Let $X(Q)$ be a rearrangement-invariant space. Then the Sobolev embedding*

$$W^1X(Q) \hookrightarrow \mathrm{BMO}(Q)$$

holds if and only if

$$X(Q) \hookrightarrow L^{n,\infty}(Q).$$

As for the inclusion of a Sobolev space into VMO, we have the following result.

Theorem 11.3. *Let $X(Q)$ be a rearrangement-invariant space. Then*

$$W^1X(Q) \subset \mathrm{VMO}(Q)$$

if and only if

$$\lim_{t\to 0_+} \tfrac{1}{t} \left\| s^{\frac{1}{n}} \chi_{(0,t)}(s) \right\|_{X'(0,1)} = 0.$$

Again, we can characterize the optimal domain partner for VMO. We need to employ the uniform absolute continuity here.

Corollary 11.4. *Let $X(Q)$ be a rearrangement-invariant space. Then the Sobolev embedding*

$$W^1X(Q) \subset \mathrm{VMO}(Q)$$

holds if and only if

$$X(Q) \subset (L^{n,\infty})_a(Q),$$

where $(L^{n,\infty})_a(Q)$ is the set of all functions having absolutely continuous norms in $L^{n,\infty}(Q)$.

The characterization of the space $(L^{n,\infty})_a(Q)$ follows from a more general result from [104] (full proof can be found in [99, Theorem 7.10.23]).

Theorem 11.5. *Assume that $1 < p < \infty$. Then*

$$(L^{p,\infty})_a(Q) = \{f \in \mathcal{M}(Q);\ \lim_{t\to 0_+} t^{\frac{1}{p}} f^{**}(t) = \lim_{t\to\infty} t^{\frac{1}{p}} f^{**}(t) = 0\}.$$

Let us turn attention to Hölder spaces and more general Morrey and Campanato spaces. A classical result due to Morrey states that if $p > n$, then any function from $W^{1,p}(Q)$ equals a.e. a Hölder continuous function. Precisely, the embedding

$$W^{1,p}(Q) \hookrightarrow C^{0,\alpha}(Q), \quad \alpha = 1 - \tfrac{p}{n}, \tag{11.1}$$

holds, where $C^{0,\alpha}(Q)$ denotes the space of Hölder continuous functions with exponent $\alpha \in (0,1]$ endowed with the seminorm

$$\|f\|_{C^{0,\alpha}(Q)} = \sup_{\substack{x,y\in Q\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

More generally, given a continuous function $\varphi : (0,\infty) \to (0,\infty)$, which will be referred to as an *admissible function*, one can consider the space $C^{0,\varphi}(Q)$ equipped with the seminorm

$$\|f\|_{C^{0,\varphi}(Q)} = \sup_{\substack{x,y\in Q\\ x\neq y}} \frac{|f(x)-f(y)|}{\varphi(|x-y|)}. \tag{11.2}$$

Obviously, $C^{0,\varphi}(Q)$ consists of (uniformly) continuous functions if $\lim_{t\to 0+} \varphi(t) = 0$. Hölder type spaces are a basic tool in various areas of analysis, including, for instance, the theory of regularity in the calculus of variations and in partial differential equations. In these and other applications, however, one is often forced to work with related function spaces defined in terms of integral, rather than pointwise, oscillation. These are the spaces of Campanato and Morrey type. In analogy with (11.2) (see [107]), given an admissible function φ, the Campanato space $L^C_\varphi(Q)$ is defined as the space of all real-valued measurable functions f on Q for which the seminorm

$$\|f\|_{L^C_\varphi(Q)} = \sup_{Q'\subset Q} \frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x) - f_{Q'}|\, dx$$

is finite, where the supremum is taken over all subcubes Q' of Q with sides parallel to those of Q. Similarly, the Morrey space $L^M_\varphi(Q)$ is defined as the space of all functions f as above such that the norm

$$\|f\|_{L^M_\varphi(Q)} = \sup_{Q'\subset Q} \frac{1}{|Q'|\varphi(|Q'|^{\frac{1}{n}})} \int_{Q'} |f(x)|\, dx$$

is finite.

In the case when $\varphi(t) = t^\alpha$ with appropriate $\alpha \in \mathbb{R}$, the spaces $L^C_\varphi(Q)$ and $L^M_\varphi(Q)$ coincide with the classical Campanato and Morrey spaces as they had been introduced originally. The relevance of Campanato spaces in the theory of regularity stems from the fact that they overlap with Hölder spaces. Indeed, a basic result tells us that $L^C_\alpha(Q) = C^{0,\alpha}(Q)$ if $0 < \alpha \leq 1$. The overlapping of Campanato and Morrey spaces is also non-empty, since $L^C_\alpha(Q) = L^M_\alpha(Q)$ if $-n \leq \alpha < 0$. In the borderline case when $\alpha = 0$, $L^M_0(Q) = L^\infty(Q)$, whereas $L^C_0(Q) = \mathrm{BMO}(Q)$, the space of functions of bounded mean oscillation over Q. We refer to [99] for a comprehensive exposition of these spaces.

We shall now turn our attention to Sobolev embeddings. We begin with Campanato spaces.

Theorem 11.6. *Let $X(Q)$ be an r.i. space and let φ be an admissible function. Set*

$$\theta(t) = \frac{t^{\frac{1}{n}}}{\varphi(t^{\frac{1}{n}})}, \quad t \in (0,\infty), \tag{11.3}$$

and let $M_\theta(Q)$ be the corresponding Marcinkiewicz space. Then the following assertions are equivalent.

(i) *A positive constant C exists such that*

$$\|u\|_{L^C_\varphi(Q)} \leq C\|\nabla u\|_{X(Q)}$$

for every $u \in W^1X(Q)$;

(ii)
$$\sup_{0<s<1} \frac{1}{s\varphi(s^{\frac{1}{n}})} \|r^{\frac{1}{n}}\chi_{(0,s)}(r)\|_{X'(0,1)} < \infty;$$

(iii)
$$X(Q) \hookrightarrow M_\theta(Q).$$

Thus, in particular, $M_\theta(Q)$ is the largest r.i. space $X(Q)$ which renders (i) *true.*

Let us next consider Morrey spaces. In this framework, we shall assume, without loss of generality, that

$$\inf_{t>0} \varphi(t) > 0. \tag{11.4}$$

Indeed, $L^M_\varphi = \{0\}$ if (11.4) is not in force.

Theorem 11.7. *Let $X(Q)$ be an r.i. space and let φ be an admissible function satisfying* (11.4)*. Let $Y_\varphi(Q)$ be the space of those functions f on Q for which the norm*

$$\|f\|_{Y_\varphi(Q)} = \sup_{0<s<1} \frac{1}{\varphi(s^{\frac{1}{n}})} \int_s^1 f^{**}(r) r^{-\frac{1}{n'}}\, dr$$

is finite, where $n' = n/(n-1)$. Then $Y_\varphi(Q)$ is an r.i. space and the following assertions are equivalent.

(i) *A positive constant C exists such that*

$$\|u\|_{L^M_\varphi(Q)} \le C\left(\|u\|_{L^1(Q)} + \|\nabla u\|_{X(Q)}\right)$$

for every $u \in W^1X(Q)$;

(ii)
$$\sup_{0<s<1} \frac{1}{\varphi(s^{\frac{1}{n}})} \|r^{-\frac{1}{n'}} \chi_{(s,1)}(r)\|_{X'(0,1)} < \infty;$$

(iii)
$$X(Q) \hookrightarrow Y_\varphi(Q).$$

Thus, in particular, $Y_\varphi(Q)$ is the largest r.i. space $X(Q)$ which renders (i) *true.*

In the following theorem we deal with embeddings into Hölder spaces.

Theorem 11.8. *Let $X(Q)$ be an r.i. space and let φ be an admissible function. Let $Z_\varphi(Q)$ be the space of those functions f on Q for which the norm*

$$\|f\|_{Z_\varphi(Q)} = \sup_{0<s<1} \frac{1}{\varphi(s^{\frac{1}{n}})} \int_0^s f^{*}(r) r^{-\frac{1}{n'}}\, dr$$

is finite. Then $Z_\varphi(Q)$ is an r.i. space and the following assertions are equivalent:

(i) *A positive constant C exists such that*

$$\|u\|_{C^{0,\varphi}(Q)} \le C\|\nabla u\|_{X(Q)}$$

for every $u \in W^1X(Q)$;

(ii)
$$\sup_{0<s<1} \frac{1}{\varphi(s^{\frac{1}{n}})} \|r^{-\frac{1}{n'}} \chi_{(0,s)}(r)\|_{X'(0,1)} < \infty;$$

(iii)
$$X(Q) \hookrightarrow Z_\varphi(Q).$$

Thus, in particular, $Z_\varphi(Q)$ is the largest r.i. space $X(Q)$ which renders (i) *true.*

As recalled above, the target spaces of the embeddings in Theorems 11.6–11.8 may coincide. The aim of the following two propositions is to combine known and new results to give a comprehensive picture about inclusion relations between generalized Campanato and Morrey spaces and between generalized Campanato and Hölder spaces. We shall assume that φ satisfies the *doubling condition*

$$C^{-1}\varphi(t) \le \varphi\left(\tfrac{t}{2}\right) \le C\varphi(t) \quad \text{for some } C > 1 \text{ and every } t > 0. \tag{11.5}$$

In many significant cases, this is not a restriction. For instance, it was shown in [90], given any non-decreasing admissible function φ, there exists another admissible function $\tilde\varphi$ satisfying (11.5) and such that $L^C_\varphi(Q) = L^C_{\tilde\varphi}(Q)$ with equivalent seminorms. The corresponding property holds, with even simpler proof, for Morrey spaces.

For technical reasons, some of our conclusions require the additional monotonicity conditions

$$t^{n-1}\varphi(t) \text{ is non-decreasing and } \frac{\varphi(t)}{t} \text{ is non-increasing on } (0,\infty), \tag{11.6}$$

which are however satisfied in customary situations.

We start with a comparison of Campanato and Morrey spaces.

Proposition 11.9. *Let φ be an admissible function satisfying* (11.5). *Let ω be the admissible function which is defined by*

$$\omega(t) = \int_t^1 \frac{\varphi(s)}{s}\, ds \tag{11.7}$$

for $t \in (0, \frac{1}{2}]$, and is constant elsewhere. Then

$$L^M_\varphi(Q) \hookrightarrow L^C_\varphi(Q) \hookrightarrow L^M_\omega(Q). \tag{11.8}$$

If moreover

$$\limsup_{t\to 0+} \frac{\omega(t)}{\varphi(t)} = \infty, \tag{11.9}$$

then the second embedding in (11.8) *is strict. Under the additional assumption* (11.6), *also the first embedding in* (11.8) *is strict and ω is optimal in the sense that whenever σ is an admissible function such that*

$$\limsup_{t\to 0+} \frac{\omega(t)}{\sigma(t)} = \infty, \tag{11.10}$$

then $L^C_\varphi(Q)$ is not embedded into $L^M_\sigma(Q)$.

We now compare Campanato and Hölder spaces.

Proposition 11.10. *Let φ be a non-decreasing admissible function satisfying* (11.5) *and such that*

$$\int_0 \frac{\varphi(t)}{t}\, dt < \infty. \tag{11.11}$$

Define

$$\psi(t) = \int_0^t \frac{\varphi(s)}{s}\, ds, \quad t \in (0,\infty).$$

Then ψ is an admissible function and

$$C^{0,\varphi}(Q) \hookrightarrow L^C_\varphi(Q) \hookrightarrow C^{0,\psi}(Q). \tag{11.12}$$

If moreover

$$\limsup_{t\to 0+} \frac{\psi(t)}{\varphi(t)} = \infty, \tag{11.13}$$

then the second embedding in (11.12) *is strict. Under the additional assumption* (11.6), *also the first embedding in* (11.12) *is strict and ψ is optimal in the sense that whenever σ is an admissible function such that*

$$\limsup_{t\to 0+} \frac{\psi(t)}{\sigma(t)} = \infty, \tag{11.14}$$

then $L^C_\varphi(Q)$ is not embedded into $C^{0,\sigma}(Q)$.

Remarks 11.11. (i) As a consequence of (11.8), we see that the spaces $L^C_\varphi(Q)$ and $L^M_\varphi(Q)$ coincide when ω is equivalent to φ. This fact is essentially contained in [107]. For φ concave, it was explicitly stated in [89, Theorem 1.1].

(ii) The fact that the second embedding in (11.12) is strict was shown in [107] under the extra assumption that $\varphi(t)/t$ is non-increasing.

Acknowledgment. I would like to express my deep gratitude to Pavel, Gábina and Petr, the main organizers of the 26th Seminar in Differential Equations, for finding the wonderful surroundings of the picturesque South Moravian village of Bořetice, for their kind invitation, for their overwhelming hospitality and above all for enabling me to spend some time within the inspirative and creative atmosphere of this great event.

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SDE 2010
XXVII Seminar in Differential Equations
Volume II

Editor:	Gabriela Holubová, Petr Nečesal
Publisher:	University of West Bohemia, Univerzitní 8, 306 14, Pilsen
Printing:	TYPOS, tiskařské závody, spol. s r.o., Pilsen

1. edition

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ISBN 978-80-261-0186-4