Gradient estimates for nonlinear elliptic equations in the plane

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Abstract: We deal with the weak solution to the Dirichlet problem

$$\begin{cases} -\text{div}\,(a(x,\nabla u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, Ω is an open bounded set in $\mathbb{R}^2$ of class $\mathcal{C}^1$, $f \in L^1(\Omega)$, and $a(x,\xi)$ is a Carathéodory function fulfilling the growth and ellipticity condition:

$$|\xi-\eta|^2 + |a(x,\xi)-a(x,\eta)|^2 \leq \left(K+\frac{1}{K}\right)\langle a(x,\xi)-a(x,\eta), \xi-\eta\rangle,$$

for some $K \geq 1$, for a.e. $x \in \Omega$, and for every $\xi, \eta \in \mathbb{R}^2$.
We establish gradient estimates for the weak solution u to problem (1) in terms of f. Let p and q such that

$$\frac{2K}{K+1} < p < 2 < q < \frac{2K}{K-1}.$$

Given an rearrangement invariant space $Y(\Omega)$, which is an interpolation space between $L^p(\Omega)$ and $L^q(\Omega)$, we exhibit the optimal (largest) rearrangement invariant space $X(\Omega)$ such that

$$\|\nabla u\|_{Y(\Omega)} \leq c\,\|f\|_{X(\Omega)}. \tag{2}$$

We also deal with a parallel problem in the class of Orlicz spaces. Namely, given an Orlicz space $L^B(\Omega)$ which is an interpolation space between $L^p(\Omega)$ and $L^q(\Omega)$, we find the optimal Orlicz space $L^A(\Omega)$ such that

$$\|\nabla u\|_{L^B(\Omega)} \leq c\,\|f\|_{L^A(\Omega)}. \tag{3}$$

As a consequence, the modulus of continuity of the solution u to problem (1) can be determined in terms of the ambient space of f. In particular, our results complement and improve several results in the literature, which deal with the case of special norms of Zygmund and Lorentz type, including [**AAS, CS, COS**].

References

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